Characterization of non stationary unitary Processes

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Outline of the talk: **Characterization of Unitary Gaussian Processes with**

1. **Uniformly continuous, independent and stationary increments**
2. **Strongly continuous, independent and stationary increments**
3. **Uniformly continuous, independent increments.**

Wiener process/Brownian motion

On the space $\Omega = C_0(\mathbb{R}_+, \mathbb{R})$ **of continuous paths starting from origin, there is a construction of probability measure** P **by Wiener. This measure is called Wiener measure and with respect which the co-ordinate function (Wiener process)** $B_t(w) := w(t)$ **satisfies:**

For any $0 \leq t_1 < t_2 \cdots t_n < \infty$, **the increments** $B_{t_1}, B_{t_2} - B_{t_1}, \cdots B_{t_n} - B_{t_{n-1}}$ **are independent Gaussian random variables with mean** 0 **and** $Var(B_t - B_s) = t - s$.

$P(\{w : w(t) - w(s) \leq x\}) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{x} e^{-\frac{y^2}{2(t-s)}} dy.$

The Wiener Process B_t **as operator in** $L^2(\Omega, P)$ $\widetilde{Q}_t \Psi(w) = B_t.\Psi(w) = w(t)\Psi(w), w \in \Omega.$

Pulling back of the Wiener Space $L^2(\Omega, P)$ to the Symmetric Fock Space $\Gamma(L^2(\mathbb{R}_+, \mathbb{C}))$ by the unitary isomorphism given below

Symmetric Fock Space $\Gamma(\mathcal{K}) := \oplus_{n\geq 0}\mathcal{K}^{\otimes n}$

Exponential vector $e(f) := \oplus_{n\geq 0}\frac{1}{\sqrt{n!}}f^{\otimes n}$, $\langle e(f), e(g)\rangle = \sum_{n\geq 0}\frac{1}{n!}\langle f, g\rangle^n = e^{\langle f, g\rangle}$

Vacuum vector $e(0) = 1 \oplus 0 \oplus 0 \oplus \cdots$, $\langle e(f), e(0)\rangle = 1$

Annihilation Process $a(t)e(f) = e(f)\int_0^t f(s)ds$

Creation Process $\langle e(f), a^\dagger(t)e(g)\rangle = \langle a(t)e(f), e(g)\rangle$

Commutation relation $[a(t), a^\dagger(s)] = t \wedge s \; 1_\Gamma$

Vacuum Expectation For $X \in \mathcal{B}(\Gamma(L^2(\mathbb{R}_+)), \mathbb{E}_0 X = \langle e(0), Xe(0)\rangle$

Unitary Isomorphism $W : \Gamma(L^2(\mathbb{R}_+)) \rightarrow L^2(P)$

$\Psi_f = We(f) := e^{\int_0^\infty f(s)dB_s - \frac{1}{2}\int_0^\infty f^2(s)ds}$ **unitarity follows from** $\mathbb{E}_P(\overline{\Psi_f}.\Psi_g) = \langle e(f), e(g)\rangle$

In this Unitary Isomorphism

$\widetilde{Q}_t = W[a^\dagger(t) + a(t)]W^* =: WQ_tW^*$ **which can be seen from**

$\langle \Psi_f, \widetilde{Q}\Psi_g\rangle = \int_0^t[\overline{f(s)} + g(s)]ds\langle e(f), e(g)\rangle = \langle e(f), Q_t e(g)\rangle$

Stochastic Evolution Consider the **Hudson-Parthasarathy type** Quantum Stochastic Differential Equation on $\mathbf{h}\otimes\Gamma(L^2(\mathbb{R}_+))$: with $H,L\in\mathcal{B}(\mathbf{h}),H$ self adjoint

$$V_{s,t}=1+\int_s^t V_{s,\lambda}La^\dagger(d\lambda)+\int_s^t V_{s,\lambda}(-L^*)a(d\lambda)+\int_s^t V_{s,\lambda}(-\frac{1}{2}L^*L+iH)d\lambda$$

In n-dimensional Brownian motion $B(t)=(B_1(t),\cdots B_n(t))$ or k-valued Brownian motion with $\mathbf{k}$ a separable Hilbert space, corresponding Fock space will be $\Gamma(L^2(\mathbb{R}_+,\mathbf{k})$. With respect to a choice of orthonormal basis $\{E_j\}_{j\geq 1}$ of $\mathbf{k}$, $a_j(t)$ and $a_j^\dagger(t)$ can be defined and shown that $[a_j(t),a_k^\dagger(s)]=\delta_{j,k}\ t\wedge s$. Here we consider the HP type Equation :

$$V_{s,t}=1+\int_s^t\sum_j V_{s,\lambda}L_j(\lambda)a_j^\dagger(d\lambda)+\int_s^t\sum_j V_{s,\lambda}(-L_j(\lambda)^*)a_j(d\lambda)+\int_s^t V_{s,\lambda}G(\lambda)d\lambda$$

with $G(\lambda),L_j(\lambda)$ be bounded measurable families of operators in $\mathcal{B}(\mathbf{h})$ and $\|\sum_j L_j(\lambda)u\|^2:=-2Re\langle u,G(\lambda)u\rangle$.

It can be seen that solution $V_{s,t}$ of above equation satisfies certain hypotheses given below.

Unitary process with independent and stationary increments

Let $\{U_{s,t} : 0 \leq s \leq t < \infty\}$ be a family of unitary operators in $\mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ satisfies

A1 Evolution For any $0 \leq r \leq s \leq t < \infty$, $U_{r,s}U_{s,t} = U_{r,t}$.

A2 Independence of increments For any $0 \leq s_i \leq t_i < \infty \ : \ i = 1,2$ such that $[s_1,t_1] \cap [s_2,t_2] = \varnothing$

(a) $U_{s_1,t_1}(u_1,v_1)$ commutes with $U^{\#}_{s_2,t_2}(u_2,v_2)$ for every $u_i, v_i \in \mathbf{h}$.

(b) For $s_1 \leq \underline{\mathbf{a}}, \underline{\mathbf{b}} \leq t_1, s_2 \leq \underline{\mathbf{q}}, \underline{\mathbf{r}} \leq t_2, \underline{\mathbf{u}}, \underline{\mathbf{v}} \in \mathbf{h}^{\otimes n}, \underline{\mathbf{p}}, \underline{\mathbf{w}} \in \mathbf{h}^{\otimes m}$

$$\langle \Omega, U^{\#}_{a_1,b_1}(u_1,v_1) \cdots U^{\#}_{a_m,b_m}(u_m,v_m) U^{\#}_{q_1,r_1}(p_1,w_1) \cdots U^{\#}_{q_n,r_n}(p_n,w_n)\Omega \rangle$$
$$= \langle \Omega, U^{\#}_{a_1,b_1}(u_1,v_1) \cdots U^{\#}_{a_m,b_m}(u_m,v_m)\Omega \rangle \langle \Omega, U^{\#}_{q_1,r_1}(p_1,w_1) \cdots U^{\#}_{q_n,r_n}(p_n,w_n)\Omega \rangle$$

A3 (Stationarity of increments) For $0 \leq s \leq t < \infty$ and $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in \mathbf{h}^{\otimes n}$

$$\langle \Omega, U^{\#}_{s,t}(u_1,v_1) \cdots U^{\#}_{s,t}(u_n,v_n)\Omega \rangle = \langle \Omega, U^{\#}_{t-s}(u_1,v_1) \cdots U^{\#}_{t-s}(u_n,v_n)\Omega \rangle.$$

B1 **(Weak continuity)**
$\lim_{t\to 0} \langle \Omega, (U_t - 1)(u,v)\Omega \rangle = 0, \forall u, v \in \mathbf{h}$.

OR

B2 **(Uniform continuity)**
$\lim_{t\to 0} \sup\{|\langle \Omega, (U_t - 1)(u,v)\Omega \rangle| : \|u\|, \ \|v\| = 1\} = 0$.

OR

B3 **(Regularity for non stationary case)**
For $\infty > t \geq s \geq 0$**,**

$$\sup\{|\langle \Omega, (U_{s,t} - 1)(u,v)\Omega \rangle| : \|u\| = \|v\| = 1\} \leq C|t-s|$$

for some positive constant C **independent of** s,t.

C1 **(Gaussian Condition)** **For any** $u_i, v_i \in \mathbf{h} : i = 1,2,3$
$\lim_{t\to 0}\frac{1}{t}\langle \Omega, (U_t^\# - 1)(u_1,v_1)(U_t^\# - 1)(u_2,v_2)(U_t^\# - 1)(u_3,v_3)\Omega\rangle = 0.$
OR

C2 **(Gaussian Condition for non stationary case)**

$$\lim_{t\downarrow s}\frac{1}{t-s}\langle \Omega, (U_{s,t}^\# - 1)(u_1,v_1)(U_{s,t}^\# - 1)(u_2,v_2)(U_{s,t}^\# - 1)(u_3,v_3)\Omega\rangle = 0.$$

D **(Minimality)**
The set $\mathcal{S} = \{U_{s_1,t_1}(u_1,v_1)\cdots U_{s_n,t_n}(u_n,v_n)\Omega : 0 \le s_1 \le t_1 \le s_2 \cdots s_n \le t_n < \infty, n \ge 1, \underline{\mathbf{u}} = \otimes_{i=1}^n u_i, \underline{\mathbf{v}} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}\}$ **is total in** $\mathcal{H}$.

Problem **Our aim here is to address the CONVERSE: given a family of unitary operators** $\{U_{s,t}\}$, **satisfying some properties listed above, on** $\mathcal{B}(\mathbf{h}\otimes\mathcal{H})$ **with a distinguish unit vector** $\Omega \in \mathcal{H}$ **is** $\{U_{s,t}\}$ **necessarily a solution of HP type equation as above?**

Root of this problem is from Schürmann's (PTRF 1990) paper where author has discussed the problem when $\mathbf{h}$ is finite dimensional and obtained the result by some co-algebraic techniques. The problem for Fock space adapted unitary process is discussed in: Journé (PTRF 1987), Hudson and Lindsay (Math. Proc. Camb. Philos. Soc. 1987), Lindsay and Wills (JFA 2000).

Main Results

(Sahu, Schürmann and Sinha: Publ. R.I.M.S 2009) For the unitary family with independent and stationary increments and Uniformly continuity :

(i)There exist a separable Hilbert space $\mathbf{k}$ with $dim(\mathbf{k}) = N$ (not necessarily finite) $H \in \mathcal{B}(\mathbf{h})$ self adjoint and $L_j \in \mathcal{B}(\mathbf{h}) : j = 1, 2, \cdots N : \sum_{j=1}^{N} L_j^* L_j$ converges strongly.
(ii) Consider the Unitary solution of HP equation
$V_{s,t} = 1 + \sum_{j=1}^{N} \int_s^t V_{s,\lambda} L_j a_j^\dagger(d\lambda)$
$+ \sum_{j=1}^{N} \int_s^t V_{s,\lambda}(-L_j^*) a_j(d\lambda) + \int_s^t V_{s,\lambda}(-\frac{1}{2}\sum_{j=1}^{N} L_j^* L_j + iH) d\lambda.$
Then there exist a unitary isomorphism $\Theta : \mathbf{h} \otimes \mathcal{H} \rightarrow \mathbf{h} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ such

that $\Theta U_{s,t} = V_{s,t}\ \Theta,\ \forall\ \ t \geq s \geq 0.$

Sahu, Sinha: AIHP 2010 **Under the Weak continuity problem is discussed.**

More generally, for non stationary unitary processes $\{U_{s,t}\}$ **with Regularity B3 and Gaussianity C2:**

(i) There exist a measurable family of separable Hilbert space $\mathbf{k}_t$ **with** $dim(\mathbf{k}_t) = d(t)$ **and** $G(t), L_j(t)$ **bounded measurable family of operators in** $\mathcal{B}(\mathbf{h})$ **such that** $\|\sum_j L_j(\lambda)u\|^2 := -2Re\langle u, G(\lambda)u\rangle$.

(ii) The Unitary solution $\{V_{s,t}\}$ **of the HP type equation in** $\mathbf{h} \otimes \Gamma(\int_{\mathbb{R}_+}^{\oplus} \mathbf{k}_t dt)$ **:**
$V_{s,t} = 1 + \sum_j \int_s^t V_{s,\lambda} L_j(\lambda) a_j^\dagger(d\lambda)$
$+ \sum_j \int_s^t V_{s,\lambda}(-L_j(\lambda)^*) a_j(d\lambda) + \int_s^t V_{s,\lambda}(G(\lambda))d\lambda$, **is unitarily equivalent to** $\{U_{s,t}\}$.

Ji, Sahu and Sinha: Characterization of unitary processes with independent increments; Submitted to "Communications on Stochastic Analysis". Available in Math arXiv.

Lines of Argument

1. Finding a $*$-algebra M and Positive definite kernel $K_s : s \geq 0$ to capture the Hilbert Space $\mathbf{k}_s$ by Kolmogorov's construction.
2. Finding $L_j(s)$ and $G(s)$
3. Establishment of unitary equivalence

Algebra M

Let $M_0 := \{(\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\varepsilon}) : \ \underline{\mathbf{u}} = \otimes_{i=1}^n u_i, \ \underline{\mathbf{v}} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}, \underline{\varepsilon} \in \mathbb{Z}_2^n, \ n \geq 1\}$. **Then the relation '$\sim$' :** $(\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\varepsilon}) \sim (\underline{\mathbf{p}}, \underline{\mathbf{w}}, \underline{\varepsilon}')$ **if** $\underline{\varepsilon} = \underline{\varepsilon}'$ **and** $|\underline{\mathbf{u}} >< \underline{\mathbf{v}}| = |\underline{\mathbf{p}} >< \underline{\mathbf{w}}| \in \mathcal{B}(\mathbf{h}^{\otimes n})$ **is an equivalence relation on** M_0. **Now consider the $*$-algebra M generated by** $M_0/\sim$ **with**

Multiplication $(\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\varepsilon}).(\underline{\mathbf{p}}, \underline{\mathbf{w}}, \underline{\varepsilon}') = (\underline{\mathbf{u}} \otimes \underline{\mathbf{p}}, \underline{\mathbf{v}} \otimes \underline{\mathbf{w}}, \underline{\varepsilon} \oplus \underline{\varepsilon}')$

Involution $(\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\varepsilon})^* = (\underleftarrow{\mathbf{v}}, \underleftarrow{\mathbf{u}}, \underline{\varepsilon}^*)$

where for $\underline{\mathbf{u}} = u_1 \otimes u_2 \cdots \otimes u_n, \underline{\varepsilon} = (\varepsilon_1, \cdots \varepsilon_n), \underline{\varepsilon}' = (\varepsilon_1', \cdots \varepsilon_m') : \quad \underline{\varepsilon} \oplus \underline{\varepsilon}' = (\varepsilon_1, \cdots \varepsilon_n, \varepsilon_1', \cdots \varepsilon_m') \in \mathbb{Z}_2^{n+m}$, **and** $\underline{\varepsilon}^* = \underline{1} + (\varepsilon_n, \cdots \varepsilon_1) \in \mathbb{Z}_2^n$ **and** $\underleftarrow{\mathbf{u}} = u_n \otimes u_{n-1} \cdots \otimes u_1$.

Correspondence $(\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\varepsilon}) \leftrightarrow U_{s,t}^{(\varepsilon_1)}(u_1, v_1) \cdots U_{s,t}^{(\varepsilon_n)}(u_n, v_n) =: U_{s,t}^{(\underline{\varepsilon})}(\underline{\mathbf{u}}, \underline{\mathbf{v}})$,
$U_{s,t}^{(\underline{\varepsilon})} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$

Positive Definite Kernel K_s on M

$$K_s\left((\underline{\mathbf{u}},\underline{\mathbf{v}},\underline{\varepsilon}),(\underline{\mathbf{w}},\underline{\mathbf{z}},\underline{\varepsilon}')\right) := \lim_{t\downarrow s}\frac{1}{t-s}\left\langle \left(U_{s,t}^{(\underline{\varepsilon})}-1\right)(\underline{\mathbf{u}},\underline{\mathbf{v}})\Omega, \left(U_{s,t}^{\underline{\varepsilon}'}-1\right)(\underline{\mathbf{p}},\underline{\mathbf{w}})\Omega\right\rangle$$

Due to Gaussianity,

$$K_s\left((\underline{\mathbf{u}},\underline{\mathbf{v}},\underline{\varepsilon}),(\underline{\mathbf{p}},\underline{\mathbf{w}},\underline{\varepsilon}')\right)$$

$$= \lim_{t\downarrow s}\frac{1}{t-s}\left\langle \left(U_{s,t}^{(\underline{\varepsilon})}-1\right)(\underline{\mathbf{u}},\underline{\mathbf{v}})\Omega, \left(U_{s,t}^{(\underline{\varepsilon}')}-1\right)(\underline{\mathbf{p}},\underline{\mathbf{w}})\Omega\right\rangle$$

$$= \sum_{1\le i\le m,\ 1\le j\le n}\ \prod_{k\neq i}\overline{\langle u_k,v_k\rangle}\prod_{l\neq j}\langle p_l,w_l\rangle$$

$$\times \lim_{t\downarrow s}\frac{1}{t-s}\left\langle (U_{s,t}-1)^{(\varepsilon_i)}(u_i,v_i)\Omega, (U_{s,t}-1)^{(\varepsilon_j')}(p_j,w_j)\Omega\right\rangle.$$

Since

$$
\begin{aligned}
&\langle (U_{s,t}-1)\,(u,v)\Omega, (U_{s,t}-1)\,(p,w)\Omega\rangle \\
&\quad = \langle U_{s,t}(u,v)\Omega, U_{s,t}(p,w)\Omega\rangle - \overline{\langle u,v\rangle}\langle p,w\rangle \\
&\qquad - \overline{\langle u,v\rangle}\,\langle \Omega, (U_{s,t}-1)\,(p,w)\Omega\rangle - \overline{\langle \Omega, (U_{s,t}-1)\,(u,v)\Omega\rangle}\langle p,w\rangle \\
&\quad = \langle p, (Z_{s,t}-1)\,(|w><v|)u\rangle - \overline{\langle u,v\rangle}\,\langle p, (T_{s,t}-1)\,w\rangle - \overline{\langle u, (T_{s,t}-1)\,v\rangle}\langle p,w\rangle,
\end{aligned}
$$

the existence of the limits on the right hand side follows from the continuity of the evolutions $\{Z_{s,t}\}, \{T_{s,t}\}$ defined by

$$
\langle u, T_{s,t}v\rangle := \langle \Omega, U_{s,t}(u,v)\Omega\rangle\,.
$$

and

$$
\langle p, Z_{s,t}(|w><v|)u\rangle := \langle U_{s,t}(u,v)\Omega, U_{s,t}(p,w)\Omega\rangle\,.
$$

K_s is given by

$$
\begin{aligned}
&K_s((u,v,\varepsilon),(p,w,\varepsilon')) \qquad (1)\\
&= (-1)^{\varepsilon+\varepsilon'} \lim_{t\downarrow s} \left\{ \left\langle p, \frac{Z_{s,t}-1}{t-s}(|w><v|)u \right\rangle - \overline{\langle u,v\rangle} \left\langle p, \frac{T_{s,t}-1}{t-s} w \right\rangle \right\}\\
&\quad - (-1)^{\varepsilon+\varepsilon'} \lim_{t\downarrow s} \overline{\left\langle u, \frac{T_{s,t}-1}{t-s} v \right\rangle} \langle p,w\rangle\\
&= (-1)^{\varepsilon+\varepsilon'} \left\{ \langle p, \mathcal{L}(s)(|w><v|)u\rangle - \overline{\langle u,v\rangle} \langle p, G(s)w\rangle - \overline{\langle u, G(s)v\rangle} \langle p,w\rangle \right\}.
\end{aligned}
$$

Thus K_s define a positive definite kernel and Kolmogorov's construction give a Hilbert space $\mathbf{k}_s$ with embedding $\eta_s(\underline{\mathbf{u}},\underline{\mathbf{v}},\underline{\varepsilon})$. Gaussianity gives that $\{\eta_s(u,v) : u,v \in \mathbf{h}\}$ is total in $\mathbf{k}_s$. Let $d(s) = dim(\mathbf{k}_s)$ and consider the basis $\{E_j(s) : j = 1,2,\cdots d(s)\}$ where $\{E_j : j \geq 1\}$ be a fixed orthonormal basis for the separable Hilbert space $l^2(N)$

Coefficient $L_j(t)$

Lemma 1. *There exists a unique measurable family* $\{L_j(t)\}$ *in* $\mathcal{B}(\mathbf{h})$ *such that* $\langle u, L_j(t)v\rangle = \langle E_j(t), \eta_t(u,v)\rangle$ *and* $\sum_{j\geq 1}\|L_j(t)u\|^2 = Re\langle u, G(t)u\rangle,\ \forall\ u \in \mathbf{h}$.

Unitary Equivalence: **Now consider the unitary solution $\{V_{s,t}\}$ of Hudson-Parthasarathy type equation, on $\mathbf{h}\otimes\Gamma(\int_{\mathbb{R}_+}^{\oplus}\mathbf{k}_s ds)$, with coefficients $L_j(t)$ and $G(t)$.**

Recall that $\mathcal{S} = \{U_{s_1,t_1}(u_1,v_1)\cdots U_{s_n,t_n}(u_n,v_n)\Omega : \quad 0 \leq s_1 \leq t_1 \leq s_2\cdots s_n \leq t_n < \infty, n \geq 1, \underline{\mathbf{u}} = \otimes_{i=1}^n u_i, \underline{\mathbf{v}} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}\}$ **is total in** $\mathcal{H}$. **Let** $\mathcal{S}' := \{V_{s_1,t_1}(u_1,v_1)\cdots V_{s_n,t_n}(u_n,v_n)\mathbf{e}(0) : 0 \leq s_1 \leq t_1 \leq s_2\cdots s_n \leq t_n < \infty, n \geq 1, \underline{\mathbf{u}}, \underline{\mathbf{v}} \in \mathbf{h}^{\otimes n}\}$.

Lemma 2. *The set* $\mathcal{S}'$ *is total in* $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$.

Now we ready to established an unitary isomorphism. Define a map $\Theta : \mathcal{H} \to \Gamma$ by setting,
$\Theta U_{s_1,t_1}(u_1,v_1)\cdots U_{s_n,t_n}(u_n,v_n)\Omega := V_{s_1,t_1}(u_1,v_1)\cdots V_{s_n,t_n}(u_n,v_n)\mathbf{e}(0), \Theta\Omega := e(0)$. **This extends to a unitary operator and $\Theta U_{s,t} = V_{s,t}\ \Theta,\ \forall\ \ t \geq s \geq 0$.**

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