Joint Extension of States of subsystems

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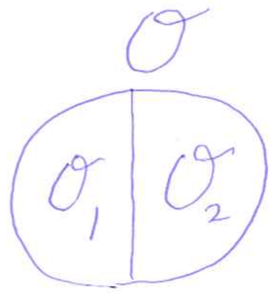
1. Introduction

- **Situation Considered**

Two Regions $\mathcal{O}_1$ and $\mathcal{O}_2$ at a distance

Global Region $\mathcal{O}$ containing $\mathcal{O}_1$ and $\mathcal{O}_2$.



A state φ of an electron in $\mathcal{O}$.

Consider the restrictions φ_1 and φ_2 of φ to $\mathcal{O}_1$ and $\mathcal{O}_2$.

- **Mathematical Formulation**

$\mathfrak{A}$: The algebra of observables for electrons in $\mathcal{O}$.

$\mathfrak{A}_1$ and $\mathfrak{A}_2$: subalgebras of $\mathfrak{A}$
describing observables localized in $\mathcal{O}_1$ and $\mathcal{O}_2$.

φ : a state of $\mathfrak{A}$.
φ_1 and φ_2 are restrictions of φ to $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

- **Problems considered**

The pair (φ_1, φ_2) arbitrary? Some restrictions?
Given the pair (φ_1, φ_2), what can we say about φ ?

We present Theorems about existence of some tight restrictions for the triple $\varphi, \varphi_1, \varphi_2$.

It is up to you how to use these results!

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2. Mathematical Model

- $\mathbb{L}$: discrete lattice (as the total space)
- observables: a_i, a_i^*, $i \in L$

satisfying

canonical anti-commutation relations

(Fermions on a lattice)

- I : a finite subset of lattice points
- $\mathfrak{A}(I)$: the algebra generated by

$$a_i, a_i^*, \quad i \in I$$

- **Even-oddness:** $\Theta \in \mathrm{Aut}(\mathfrak{A})$, $\Theta^2 = 1$, $\Theta(a_i) = -a_i$ $i \in I$

- $\mathfrak{A}_{\pm} = \{A \in \mathfrak{A} : \Theta(A) = \pm A\}$

$$\mathfrak{A} = \mathfrak{A}_+ + \mathfrak{A}_-.$$

3. Product Extensions

Definition A state φ of $\mathfrak{A}(\bigvee_i I_i)$ is called a product state of states φ_i of $\mathfrak{A}(I_i)$ for mutually disjoint regions $I_1, I_2,$, if

$$\varphi(A_1 A_2 \dots A_k) = \prod_{i=1}^{k} \varphi_i(A_i)$$

for any $A_i \in \mathfrak{A}(I_i)$ and for all k.

Definition. A state φ of $\mathfrak{A}$ or $\mathfrak{A}(I_i)$ is said to be even if it is Θ-Invariant:

$\boxed{\varphi(\Theta A) = \varphi(A)}$ for any $A \in \mathfrak{A}$ or any $A \in \mathfrak{A}(I_i)$.

Theorem 1 Let $I_1, I_2 \dots$ be an arbitrary (finite or infinite) number of mutually disjoint subsets of $\mathbb{L}$.

(1) A product state of states φ_i of $\mathfrak{A}(I_i)$, i=1,2,... exists if and only if all φ_i (except for at most one) are even.
It is unique if it exists.
It is even if and only if all φ_i are even.

(2) Suppose that all φ_i are pure states. If their joint extension exists, then all states except at most one must be even.
In that case, the joint extension is uniquely the product extension and is pure.

In (2) above, the product property of the extension is not assumed but is derived.

The purity of the component states does not follow from that of their joint extension.

For product extension, however, we have the following result.

Theorem 2. Let φ be the product extension of states φ_i of $\mathfrak{A}(I_i)$ with mutually disjoint I_i. Assume that all φ_i except for φ_1 are even.

(1) φ_1 is pure if φ is pure.

(2) Assume that π_{φ_1} and $\pi_{\varphi_1 \Theta}$ are not disjoint. Then φ is pure if and only if all φ_i are pure.

Remark. If I_1 is finite, the assumption of (2) above holds and hence the conclusion follows automatically.

In the case not covered by Theorem 2, the following result gives a complete analysis, if we take $\bigcup_{i \geqq 2} I_i$ in Theorem 2 as one subset of $\mathbb{L}$.

Theorem 3 Let φ be the product state extension of states φ_1 and φ_2 of $\mathfrak{A}(I_1)$ and $\mathfrak{A}(I_2)$ with disjoint I_1 and I_2, where φ_2 is even and φ_1 is such that π_{φ_1} and $\pi_{\varphi_1 \Theta}$ are disjoint.

(1) φ is pure if and only if φ_1 and the restriction φ_{2+} of φ_2 to $\mathfrak{A}(I_2)_+$ are both pure.

(2) Assume that φ is pure. φ_2 is not pure if and only if

$$\varphi_2 = \frac{1}{2}(\hat{\varphi}_2 + \hat{\varphi}_2 \Theta)$$

where $\hat{\varphi}_2$ is pure and $\pi_{\hat{\varphi}_2}$ and $\pi_{\hat{\varphi}_2 \Theta}$ are disjoint

4. A pair of Pure and General States for Two Subsystems.

- **Background**: A result for two general states

Theorem 4 Let φ_1 and φ_2 be states of two subsystems.

Assume that the GNS representations π_{φ_1} and $\pi_{\varphi_1} \textcircled{+}$ are disjoint.

(Complete opposite of even φ_1)

⇒ Then a joint extension of φ_1 and φ_2 exists, if and only if φ_2 is even.

[Consequence: For some type of a pair of states, (for two subsystems of Fermions) no joint extension exists!]

- Detailed results when one of the two subsystem states is pure:

Definition:

(1) π: a representation of a C^*-algebra $\mathfrak{A}$ in a Hilbert space $\mathcal{H}$.

$\mathcal{H}$ contains vectors Φ and Ψ such that

$$\varphi(A) = (\Phi, \pi(A)\Phi)$$
$$\psi(A) = (\Psi, \pi(A)\Psi) \qquad A \in \mathfrak{A}$$

(The inner product is linear in the second vector.)

The transition probability between φ and ψ is

$$P(\varphi, \psi) = \sup |(\Phi, \Psi)|^2$$

(The supremum "sup" is taken over all $\mathcal{H}$, π, Φ, Ψ.)

Definition

(2) $p(\varphi) = P(\varphi, \varphi \textcircled{H})^{1/2}$

If φ is pure, then there are only two alternatives:

(α) φ and $\varphi \textcircled{H}$ are mutually disjoint $\Leftrightarrow$ $p(\varphi) = 0$

(β) φ and ~~$p(\textcircled{H})$~~ $\varphi \textcircled{H}$ are unitarily equivalent

$\Leftrightarrow p(\varphi) = |(\Omega_\varphi, \Omega_{\varphi \textcircled{H}})|$.

The quantity $p(\varphi)$ has the range $[0, 1]$.

Even in the case (β), $p(\varphi)$ can take the value 0.

On the other hand, $p(\varphi) = 1$ if and only if $\varphi = \varphi \textcircled{H}$.

Definition

(1) For two states φ and ψ, define

$$\lambda(\varphi, \psi) = \sup \{\lambda \in \mathbb{R} \mid \varphi - \lambda \psi \geq 0\}$$

$$\Leftrightarrow \varphi(A^*A) - \lambda \psi(AA^*) \geq 0 \ (A \in$$

(2) For a state φ, define

$$\lambda(\varphi) = \lambda(\varphi, \varphi \textcircled{H}).$$

The range of λ is also $[0, 1]$.

If $\lambda(\varphi) = 1$, then $\varphi = \varphi \textcircled{H}$.

A complete answer for a joint extension of states φ_1 and φ_2 of two subsystems when one of φ_1 and φ_2 is pure is given by the following Theorem.

Theorem 5

(1) A joint extension φ of φ_1 and φ_2 exists

$$\Leftrightarrow \boxed{\lambda(\varphi_2) \geqq \frac{1 - p(\varphi_1)}{1 + p(\varphi_1)}}$$

(2) If the above condition holds and $\boxed{p(\varphi_1) \neq 0}$, then the joint extension φ is unique and satisfies

$$\boxed{\varphi(A_1 A_2) = \varphi_1(A_1)\, \varphi_2(A_{2+}) + \frac{1}{p(\varphi_1)}\, f(A_1)\, \varphi_2(A_{2-})}$$

⟨the even part of A_2⟩ ⟨the odd part of A_2⟩

$$f(A_1) = (\Omega_{\varphi_1},\ \pi_{\varphi_1}(A_1)\, u_1\, \Omega_{\varphi_1})$$

for $A_1 \in \mathfrak{A}_1$, $A_2 = A_{2+} + A_{2-}$, $A_{2\pm} \in \mathfrak{A}_{2\pm}$ and

for u_1: Ⓗ-implementing unitary on GNS repres. $(\pi_{\varphi_1}, \mathcal{H}_{\varphi_1})$ of which satisfies $(\Omega_{\varphi_1}, u_1 \Omega_{\varphi_1}) > 0$.

← fix the phase of u_1

(3) If $p(\varphi_1) = 0$, the inequality in (1) is equivalent to $\varphi_2 = \varphi_2$Ⓗ and at least the product state extension holds.

(4) Assume that $p(\varphi_1) = 0$ and φ_2 is even. Then there exists a joint extension of φ_1 and φ_2 other than the unique product extension if and only if φ_1 and φ_2 satisfy the following two conditions:

(4-1) π_{φ_1} and π_{φ_1}Ⓗ are unitarity equivalent.

(4-2) There exists a state $\tilde{\varphi}_2$ of $\mathfrak{A}_2$ such that

$\tilde{\varphi}_2 \neq \tilde{\varphi}_2$Ⓗ and $\varphi_2 = \frac{1}{2}(\tilde{\varphi}_2 + \tilde{\varphi}_2$Ⓗ$)$.

(5) If all the conditions in (4), including

$p(\varphi_1) = 0$ and φ_2 is even, are satisfied,

then, for every $\tilde{\varphi}_2$, there exists a joint extension

φ of φ_1 and φ_2 satisfying

$$\boxed{\varphi(A_1 A_2) = \varphi_1(A_1)\varphi_2(A_{2+}) + (\Omega_{\varphi_1}, \pi_{\varphi_1}(A_1) u_1 \Omega_{\varphi_1}) \tilde{\varphi}_2(A_{2-}}$$

These extensions together with the product state extensio

exhaust all joint extensions of φ_1 and φ_2.

5. Examples

When states of subsystems are mixtures,

there are many joint extensions of given subalgebra sta

even for Fermion systems.

We give two examples in the case of two subsystems

$\mathfrak{A}_1$ and $\mathfrak{A}_2$ which can be full matrix algebras

(of finite dimensions).

Example 1 Let ρ be an invertible density matrix of a state

of the total system $\mathfrak{A} = \mathfrak{A}_{12}$ satisfying

$$\rho \geq \lambda \mathbb{1} \quad \text{for some } \lambda > 0.$$

Let $x = x^* \in \mathfrak{A}_{1-}$ and $y = y^* \in \mathfrak{A}_{2-}$ satisfying

$$\|x\| \, \|y\| \leq \lambda .$$

Let

$$\varphi_1(A_1) = \mathrm{Tr}(\rho A_1), \quad A_1 \in \mathfrak{A}_1,$$

$$\varphi_2(A_2) = \mathrm{Tr}(\rho A_2), \quad A_2 \in \mathfrak{A}_2.$$

Then

$$\varphi_{\rho'}(A) = \mathrm{Tr}(\rho' A), \quad \boxed{\rho' = \rho + ixy}$$

is a state of $\mathfrak{A} = \mathfrak{A}_{12}$ and a joint extension of φ_1 and φ_2

for any choice of x and y (satisfying above conditions).

Example 2 Let φ and ψ be states of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with decompositions

$$\varphi = \lambda \varphi_1 + (1-\lambda)\varphi_2, \qquad \psi = \mu \psi_1 + (1-\mu)\psi_2$$

where

$0 < \lambda < 1$, $0 < \mu < 1$, φ_1 and φ_2 are even but ψ_1 and ψ_2 are arbitrary.

Let $\varphi_i \psi_j$ be the product states.

Then

$$\boxed{\begin{aligned} \chi = (\lambda\mu + \kappa)\varphi_1\psi_2 &+ (\lambda(1-\mu) - \kappa)\varphi_1\psi_2 \\ &+ ((1-\lambda)\mu - \kappa)\varphi_2\psi_1 + ((1-\lambda)(1-\mu) + \kappa)\varphi_2\psi_2 \end{aligned}},$$

is a one parameter family of joint extension of φ and ψ with the parameter κ satisfying

$$\boxed{-\min(\lambda\mu, (1-\lambda)(1-\mu)) \leq \kappa \leq \min((1-\lambda)\mu, \lambda(1-\mu))}.$$