Recent results in the theory of twin building lattices

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General subject matter

We introduce a class of finitely generated groups – Kac-Moody groups over finite fields – acting on some non-classical buildings. Then we mention what the viewpoint of geometric group theory enables to say. Namely:

- **Simplicity:** Kac-Moody groups provide a wide class of finitely presented simple groups (Caprace-Rémy).
- **Rigidity:** they enjoy strong rigidity properties, of the type "higher-rank vs hyperbolic spaces" (Caprace-Rémy).
- **Amenability:** they admit amenable actions on explicit compact spaces (Caprace-Lécureux, Lécureux).
- **Quasi-morphisms:** existence of non standard quasi-morphisms is well-understood (Caprace-Fujiwara).
- **Quasi-isometry:** these groups provide infinitely many quasi-isometry classes of simple groups (Caprace-Rémy).

Plan of the talk

1. Kac-Moody groups and their buildings
2. Simplicity
3. Rigidity
4. Amenability
5. Quasi-morphisms
6. Quasi-isometry

Kac-Moody groups

- Kac-Moody groups were constructed by J. Tits to generalize algebraic groups. They share many combinatorial properties with them: they have a (twin) BN-pair structure.
- They are defined by a presentation generalizing the generators and relations of SL_n (using elementary unipotent matrices).
- The defining data for a Kac-Moody group are a field $\mathbf{K}$ and a *generalized Cartan matrix*, i.e., an integral matrix $A = [A_{s,t}]_{s,t \in S}$ such that $A_{s,s} = 2$ for all $s \in S$ and $A_{s,t} \leqslant 0$ for $s \neq t$, with $A_{s,t} = 0$ if and only if $A_{t,s} = 0$.

Example

The standard example of such a group is $\Lambda = \mathbf{G}\big(\mathbf{K}[t, t^{-1}]\big)$ for $\mathbf{G}$ a simple matrix group over a field $\mathbf{K}$.

Buildings provided by Kac-Moody theory

The geometric counterpart to the BN-pair combinatorics is:

Fact

(i) *Any Kac-Moody group* Λ *naturally acts on the product* $X_- \times X_+$ *of two isomorphic buildings* $X_\pm$*, say of Weyl group* W*.*

(ii) *The explicit rule for* W *consists in deducing the Coxeter matrix* $[M_{s,t}]_{s,t\in S}$ *from* $A = [A_{s,t}]_{s,t\in S}$*. Precisely:* $M_{s,t} = 2$, 3, 4, 6 *or* ∞ *according to whether the product* $A_{s,t} \cdot A_{t,s}$ *is* 0, 1, 2, 3 *or is* $\geqslant 4$*, respectively.*

Reading backwards $[A_{s,t}]_{s,t\in S} \mapsto [M_{s,t}]_{s,t\in S}$, we can produce buildings (with nice group actions) provided the Weyl group has Coxeter exponents in $\{2; 3; 4; 6; \infty\}$.

Affine and exotic buildings

- The case of affine buildings corresponds exactly to the previous examples $\Lambda = \mathbf{G}\big(\mathbf{K}[t, t^{-1}]\big)$, with a concrete matrix interpretation. We say then that A and Λ are of *affine* type.
- When W is a Fuchsian group, e.g. W is generated by a right-angled hyperbolic polygon or by a regular triangle of angle $\frac{\pi}{4}$ or $\frac{\pi}{6}$; then $X_\pm$ carries a negatively curved metric.

These are only examples; the general case is a mixture.

Fact

(i) *Any Kac-Moody group Λ over any finite field is finitely generated.*

(ii) *The associated buildings $X_\pm$, for a suitable non-positively curved realization, are locally finite.*

Computation of covolume

From now on, Λ denotes a Kac-Moody group defined by a generalized Cartan matrix $A = [A_{s,t}]_{s,t \in S}$ and a *finite* field $\mathbf{F}_q$. The full automorphism groups $\mathrm{Aut}(X_\pm)$ are thus locally compact, and as such admit Haar measures.

Theorem (B. R., 1999)

Assume the Weyl group W of Λ is infinite and denote by $W(t) = \sum_{w \in W} t^{\ell(w)}$ its growth series. If $W(\frac{1}{q}) < \infty$, then Λ is a lattice of $X_+ \times X_-$; it is never cocompact.

- When $\Lambda = \mathbf{G}\big(\mathbf{F}_q[t, t^{-1}]\big)$, the condition $W(\frac{1}{q}) < \infty$ is empty since the virtually abelian group W has polynomial growth.
- The diagonal Λ-action on $X_+ \times X_-$ is always discrete and the real number $W(\frac{1}{q})$ is a covolume for a suitable normalization.

Simple groups

The fact that finitely generated lattices can be seen as lattices of some reasonable geometry is the starting point to prove the following simplicity result.

Theorem (P.-E. Caprace and B.R., 2009)

Let Λ be a Kac-Moody group defined over the finite field $\mathbf{F}_q$. Assume that the Weyl group W is infinite and irreducible and that $W(\frac{1}{q}) < \infty$. Then Λ is simple (modulo its finite center) whenever the buildings $X_\pm$ are not Euclidean.

- So whenever Λ has no obvious matrix interpretation, it is a simple finitely generated group
  (and there is a geometric explanation for this).
- There exist *infinitely many* matrices A such that Λ is a *finitely presented, Kazhdan, simple group* for $q \gg 1$.

Strategy of proof

- The idea is first to see Kac-Moody lattices as analogues of lattices in Lie groups to rule out infinite quotients, and finally to stand by decisive differences to rule out finite quotients too.
- The analogy part follows Margulis' strategy for the normal subgroup property; it shows that any normal subgroup of a Kac-Moody lattice has finite index (thanks to a general criterion by Bader-Shalom).
- What goes wrong for simplicity? The affine example $\Lambda = \mathbf{G}\big(\mathbf{F}_q[t, t^{-1}]\big)$ has a lot of finite quotients.
- This is where we use non-affineness: a strengthening of Tits' alternative for Coxeter groups (Margulis-Noskov-Vinberg) implies that non-affine Coxeter groups are "weakly hyperbolic". This is what we combine with a trick on infinite root systems.

Kazhdan and amenable groups, non-positive curvature

- A group is said to have (Kazhdan's) *property* (T) if it fixes a point whenever it acts by affine isometries on a separable Hilbert space. It is said to be *amenable* if, whenever it acts on a compact metrizable space X, it fixes a probability measure on X. There are many other equivalent formulations for each of these properties. The combination of them implies the compactness of the group.
- A metric space is called *geodesic* if any two points are connected by a continuous path parametrized by distance. Such a space is called $\mathrm{CAT}(0)$ if any geodesic triangle is at least as thin as in $\mathbf{R}^2$, meaning that median lengths are shorter than in the Euclidean plane. The notion of a $\mathrm{CAT}(-1)$-*space* is defined accordingly, replacing $\mathbf{R}^2$ by the hyperbolic plane.

Why caring about (T) for simple groups?

- By Peter-Weyl's theorem, *a finitely generated group is residually finite if, and only if, it embeds abstractly in a compact group*. Therefore a finitely generated simple group Γ has trivial homomorphic image in any compact group. So if Γ acts on a locally finite complex with a global fixed point, the action is actually trivial.
- By Bruhat-Tits fixed point theorem, this implies that if Γ acts non-trivially on a $\mathrm{CAT}(0)$ complete metric space, then any orbit is unbounded.
- Now let Y be a proper $\mathrm{CAT}(-1)$-space with $\mathrm{Isom}(Y)$ acting cocompactly. The stabilizer of any $\xi \in \partial_\infty Y$ is amenable (Burger-Mozes). Therefore a non-trivial action of a finitely generated *Kazhdan* simple group Γ on Y has no global fixed point in the compactification $Y \cup \partial_\infty Y$.

Super-rigidity

- This is a well-known phenomenon: there are many results disproving the existence of actions of higher-rank lattices, e.g. $\mathrm{SL}_n(\mathbf{Z})$ for $n \geqslant 3$, on the circle ($=$ boundary of $\mathbb{H}^2_{\mathbf{R}}$).
- What makes a Kac-Moody lattice being of higher-rank? In the non-affine Kac-Moody case, Kazhdan's property (T) and the existence of flats of dimension $\geqslant 2$ in the buildings are independent conditions.

Theorem (P.-E. Caprace and B.R., 2009)

Let Λ be a simple Kac-Moody lattice and let Y be a proper $\mathrm{CAT}(-1)$-space with cocompact isometry group. If the buildings $X_\pm$ of Λ contain flat subspaces of dimension $\geqslant 2$ and if Λ is Kazhdan, then the group Λ has no nontrivial action by isometries on Y.

Compactifications and amenable groups

Given a simple Lie real group like $\mathrm{SL}_n(\mathbf{R})$, a certain (Satake) compactification $\overline{X}$ of the associated symmetric space $X = G/K$ provides a geometric parametrization – up to finite index – of maximal amenable subgroups in G (Moore): any point stabilizer in G is an amenable subgroup; conversely, any amenable subgroup of G has a finite index subgroup stabilizing a point in $\overline{X}$.

Theorem (P.-E. Caprace and J. Lécureux, 2010)

Any locally finite building X admits a compactification providing the same classification for amenable subgroups in $G = \mathrm{Isom}(X)$.

This compactification is not the one given by asymptotic classes of geodesic rays; it is related to the compact space of closed subgroups of G and to the combinatorics of infinite root systems.

Amenable actions

Roughly speaking, a G-action on a space S is called *amenable* if there is a sequence of maps $\{\mu_n : S \to \mathcal{M}^1(G)\}_{n \geqslant 0}$ (to the probability measures on G) such that $\lim_{n\to\infty} \parallel \mu_n(g.x) - g_*\mu_n(x) \parallel = 0$ uniformly on compact subsets of $G \times X$. In the previous situation, the group $G = \mathrm{Isom}(X)$ itself is not amenable in general, but we have the following.

Theorem (J. Lécureux, 2010)

For any building X, any proper action by a locally compact group on the above compactification $\overline{X}$ is amenable.

Admitting an amenable action on a compact space is an important property in analytic group theory (Novikov conjecture and others). It also provides theoretical resolutions in bounded cohomology and boundary maps in rigidity theory.

Definition of quasi-homomorphisms

- The set of all quasi-characters of G is denoted by $\mathrm{QH}(G)$.
- The set of non-trivial quasi-characters is by definition
$$\widetilde{\mathrm{QH}(G)} = \frac{\mathrm{QH}(G)}{\mathrm{Hom}(G,\mathbf{R}) \oplus \ell^\infty(G)}.$$
- Quasi-homomorphisms are related to rigidity questions; higher-rank lattices in Lie groups don't have non-trivial quasi-characters (Burger-Monod).
- As we saw for rigidity, it is not clear what to require to consider that a building is of higher rank. The next result shows that many buildings are not of higher-rank with respect to quasi-homomorphims.

Existence of non-standard quasi-homomorphisms

Theorem (P.-E. Caprace and K. Fujiwara, 2010)

Let (W, S) be an infinite, irreducible, non-affine Coxeter system and let X be a building of type (W, S). Let G be a group acting on X by automorphisms so that at least one of the following conditions is satisfied:

(i) *The G-action on X is Weyl-transitive.*

(ii) *For some apartment $\mathbb{A} \subset X$, the stabilizer $\mathrm{Stab}_G(\mathbb{A})$ acts cocompactly on $\mathbb{A}$.*

Then $\widetilde{\mathrm{QH}(G)}$ is infinite-dimensional.

Combined with Kac-Moody theory, this implies that, up to isomorphism, there exist infinitely many finitely presented simple groups of strictly positive stable commutator length.

Definition of Cayley graphs and quasi-isometry

- Let Γ be a group generated by a finite subset S such that $S = S^{-1}$. The *Cayley graph* of (Γ, S) is the graph whose vertices are the elements of Γ, where we decide that $\gamma, \gamma' \in \Gamma$ are connected if there is $s \in S$ such that $\gamma = \gamma' s$.
- M. Gromov suggested in the 80ies to classify finitely generated groups via their Cayley graphs, up to the notion of quasi-isometry. Two metric spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there is a map $f : X \to Y$ such that: there exist $C \geqslant 1$ and $D \geqslant 0$ such that for any $x, x' \in X$ we have

$$\frac{1}{C} \cdot d_X(x, x') - D \leqslant d_Y(f(x), f(x')) \leqslant C \cdot d_X(x, x') + D$$

and for any $y \in Y$ there is $x \in X$ such that $d_Y(y, f(x)) \leqslant D$, i.e., (X, d_X) and (Y, d_Y) are "bilipschitz equivalent up to additive constant".

Infinitely many quasi-isometry classes of simple groups

Let G be a locally compact group admitting a finitely generated lattice Γ. This implies that G admits a compact generating subset, say $\widehat{\Sigma}$; we denote by $d_{\widehat{\Sigma}}$ the word metric associated with $\widehat{\Sigma}$. Similarly, we fix a finite generating set Σ for Γ and denote by d_Σ the associated word metric. The lattice Γ is called *undistorted* in G if d_Σ is quasi-isometric to the restriction of $d_{\widehat{\Sigma}}$ to Γ. This amounts to saying that the inclusion of Γ in G is a quasi-isometric embedding from (Γ, d_Σ) to $(G, d_{\widehat{\Sigma}})$.

Theorem (P.-E. Caprace and B.R., 2010)

Any twin building lattice $\Gamma < G_+ \times G_-$ is undistorted.

Again, combined with simplicity results from Kac-Moody theory, this implies that there exist infinitely many pairwise non-quasi-isometric finitely presented simple groups.