Symmetric chains, Gelfand-Tsetlin (GZ) chains,

and the Hypercube

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Outline of Talk

I. Problem: Explicit block diagonalization of the Terwilliger algebra of the hypercube.

II. (Very) Brief comments on coding theoretic motivation for the problem.

III. Sketch of (new) solution to the problem.

IV. The **linear de Bruijn, Tengbergen, Kruyswijk (BTK)** algorithm: statement of Main Theorems MT1, MT2, and MT3.

$B(n) =$ Set of all subsets of $\{1, 2, \ldots, n\}$, so $|B(n)| = 2^n$

$V(B(n)) =$ Complex vector space with $B(n)$ as basis

Terwilliger algebra of the Hypercube

$$\begin{aligned} \mathcal{T}_n &= \text{Commutant of the } S_n \text{ action on } V(B(n)) \\ &= \mathrm{End}_{S_n}(V(B(n))) \end{aligned}$$

What is the dimension of $\mathcal{T}_n$?

Think of elements of $V(B(n))$ as column vectors of size 2^n, with

coordinates indexed by $B(n)$.

Represent $f \in \mathrm{End}(V(B(n))$ (in the standard basis $B(n)$) as a $B(n) \times B(n)$

matrix M_f. Entry in row X, col Y of M_f is denoted by $M_f(X, Y)$.

Then it is easy to see that

$f : V(B(n)) \to V(B(n))$ **is S_n-linear iff**

$M_f(\pi(X), \pi(Y)) = M_f(X, Y), \;\; X, Y \in B(n), \pi \in S_n$, i.e.,

M_f **is constant on the orbits of the S_n-action on** $B(n) \times B(n)$.

(X, Y) **and** (X', Y') **are in the same** S_n**-orbit iff**

$|X| = |X'|, \;\; |Y| = |Y'|, \;\; |X \cap Y| = |X' \cap Y'|$

Dimension of $\mathcal{T}_n$

$$M_{i,j}^t(X, Y) = \begin{cases} 1 & \text{if } |X| = i, \; |Y| = j, \; |X \cap Y| = t \\ 0 & \text{otherwise} \end{cases}$$

$$\{M_{i,j}^t \mid i - t + t + j - t \leq n, \; i - t, t, j - t \geq 0\}$$

is a basis of $\mathcal{T}_n$ and its cardinality is $\binom{n+3}{3}$.

$\mathcal{T}_n$ *is a* C^**-algebra, so it has a* **Block Diagonalization:** There exists a $B(n) \times S$ unitary N, for some set S of size 2^n, and positive integers $p_0, q_0, \ldots, p_m, q_m$ such that $N^* \mathcal{T}_n N$ is equal to the set of all $S \times S$ block-diagonal matrices

$$\begin{pmatrix} C_0 & 0 & \ldots & 0 \\ 0 & C_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_m \end{pmatrix}$$

where each C_k is a block-doagonal matrix with q_k repeated, identical blocks of order p_k

$$C_k = \begin{pmatrix} B_k & 0 & \dots & 0 \\ 0 & B_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{pmatrix}$$

Thus $p_0^2 + \cdots + p_m^2 = \dim(\mathcal{T}_n)$ and $p_0 q_0 + \cdots + p_m q_m = 2^n$.

By dropping duplicate blocks we get a positive semidefinitenes preserving

C^*-algebra isomorphism

$$\Phi : \mathcal{T}_n \cong \bigoplus_{k=0}^{m} \mathbf{Mat}(p_k \times p_k)$$

Explicit Block Diagonalization (EBD)

$$\Phi\left(\sum_{r,s,t=0}^{n} x_{r,s}^{t} M_{r,s}^{t}\right) = (N_0, \ldots, N_m),$$

Determine the entries of N_k?

Remark The decomposition of $V(B(n))$ into S_n-irreps. is classical, giving

$m = \lfloor n/2 \rfloor,\ p_k = n - 2k + 1,\ q_k = \binom{n}{k} - \binom{n}{k-1},\ 0 \le k \le m.$

PART II

Polynomial time computable upper bounds on binary code size

$\emptyset \subset C \subset B(n)$ (proper subset) **Binary Code**

$d(X, Y) = |X \Delta Y|$ **Hamming distance** of $X, Y \in C$

$\mathbf{A(n, d)}$ = maximum size of a binary code of length n and minimum distance among distinct elements at least d.

Computing $A(n, d)$ is NP-Hard, so the focus is on lower and upper bounds for $A(n, d)$. Particularly interesting mathematically are algorithms that compute upper bounds on $A(n, d)$ in time polynomial in n. These are based on the following two steps.

(i) Upper bounding binary code size by the optimal value of an exponential size semidefinite program (SDP).

(ii) Reducing the semidefinite program to polynomial size by explicit block diagonalization of the commutants of certain group actions.

The following three results fall in this framework.

Set

$H_n =$ **all distance preserving automorphisms of** $B(n)$**, so** $|H_n| = 2^n n!$

Delsarte Bound (1973) (based on word pairs): needs EBD of the **Bose-Mesner algebra** $\text{End}_{H_n}(V(B(n)))$ of the hypercube. Since the Bose-Mesner algebra is commutative the resulting SDP is a LP.

Two recent breakthroughs in this subject are

Schrijver Bound (2005) (based on word triples): needs EBD of $\text{End}_{S_n}(V(B(n))) = \mathcal{T}_n$.

Gijswijt, Mittelmann, Schrijver Bound (preprint) (based on word quadruples): needs EBD of $\text{End}_{H_n}(V(B(n) \times B(n)))$. The authors give an ingenious reduction to EBD of $\mathcal{T}_n$.

PART III

EBD of $\mathcal{T}_n$

Lex Schrijver (2005) gave an elementary linear algebraic proof of EBD.

Then **Frank Vallentin (2009)** gave a representation theoretic proof based on classical work of **Charles Dunkl**.

We give a very simple proof (in 5 steps) that uses only standard, elementary results in representation theory. This proof also motivates the **linear BTK algorithm**, which

- gives an elementary **constructive proof** of EBD, meaning that the conjugating unitary N is also written down explicitly.
- gives a **representation theoretic interpretation** to N.

Step 1 Binomial Inversion

$M^s_{l,s} M^s_{s,k} = \sum_{p=0}^{n} \binom{p}{s} M^p_{l,k}, \;\; 0 \leq s \leq n$

since the entry of the lhs in row X, col Y with $|X| = l, |Y| = k$ is equal to the number of common subsets of X and Y of size s. By binomial inversion $M^s_{l,k} = \sum_{p=0}^{n} (-1)^{p-s} \binom{p}{s} M^p_{l,p} M^p_{p,k}, \;\; 0 \leq s \leq n$

Meaning of this equation: *Since* $M^p_{p,k} = (M^p_{k,p})^*$ *and* $\Phi(M^p_{p,k}) = (\Phi(M^p_{k,p}))^*$ *all the images under* Φ *can be written down once we know the images of* $M^p_{k,p}$. *Now* $M^p_{k,p}$ *is the inclusion matrix of p-subsets vs. k-subsets. This leads us to the up operator.*

The **up operator on the hypercube** $U : V(B(n)) \to V(B(n))$ is defined,

for $X \in B(n)$, by

$U(X) = \sum Y$**, where the sum is over all** $Y \supseteq X$ **with** $|Y| = |X| + 1$**.**

$B(n)_i =$ set of all i-element subsets of$\{1, \ldots, n\}$, $0 \leq i \leq n$. So

$V(B(n)) = V(B(n)_0) \oplus V(B(n)_1) \oplus \cdots \oplus V(B(n)_n)$ (direct sum).

An element $v \in V(B(n))$ *is* homogeneous *if* $v \in V(B(n)_i)$ *for some* i*, and*

we write $r(v) = i$*.*

A **symmetric Jordan chain** (SJC) in $V(B(n))$ is a sequence

$v = (v_1, \ldots, v_h)$ of nonzero homogeneous elements of $V(B(n))$ such that

- $U(v_{i-1}) = v_i$, for $i = 2, \ldots h$, and $U(v_h) = 0$
- v is symmetric, i.e., $r(v_1) + r(v_h) = n$, if $h \geq 2$, or else $2r(v_1) = n$, if $h = 1$.

A **symmetric Jordan basis** (SJB) of $V(B(n))$ is a basis of $V(B(n))$ consisting of a disjoint union of SJC's.

Let $\langle , \rangle$ denote the standard inner product on $V(B(n))$, i.e.,

$\langle X, Y \rangle = \delta(X, Y)$, (Kronecker delta) for $X, Y \in B(n)$. The *length*

$\sqrt{\langle v, v \rangle}$ of $v \in V(B(n))$ is denoted $\| v \|$.

Theorem 1 *There exists an SJB $J(n)$ of $V(B(n))$ such that*

(i) The elements of $J(n)$ are orthogonal with respect to $\langle , \rangle$.

(ii) **(Singular Values)** *Let $0 \leq k \leq \lfloor n/2 \rfloor$ and let $(x_k, \ldots, x_{n-k})$ be any SJC in $J(n)$ starting at rank k and ending at rank $n-k$ (there are $\binom{n}{k} - \binom{n}{k-1}$ of these). Then*

$$\frac{\| x_{u+1} \|}{\| x_u \|} = \sqrt{(u+1-k)(n-k-u)}$$

Step 2 $\mathrm{sl}(2,\mathbb{C})$ action on $V(B(n))$: there exists an SJB

Down operator D on $V(B(n))$ - analogous to the up operator.

Define the operator H on $V(B(n))$ by $H(v_i) = (2i - n)v_i$,

$v_i \in V(B(n)_i),\ i = 0, 1, \ldots, n.$

Easy to check that $[H, U] = 2U$, $[H, D] = -2D$, and $[U, D] = H$.

Thus the linear map $\mathrm{sl}(2,\mathbb{C}) \to \mathrm{gl}(V(B(n)))$ given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto U,\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto D,\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H$$

is a representation of $\mathrm{sl}(2,\mathbb{C})$.

Let $W \subseteq V(B(n))$ be an $l+1$ dimensional irreducible $sl(2,\mathbb{C})$-submodule.

Then $\mathrm{sl}(2,\mathbb{C})$ theory gives a basis of W such that the eigenvalues of H on this basis are $-l, -l+2, \ldots, l-2, l$. So the elements of this basis are symmetrically located about the middle. Write this basis as $(x_k, \ldots, x_{n-k})$, where $r(x_i) = i$, for all i. Put $x_{k-1} = x_{n-k+1} = 0$. We have

$$U(x_u) = x_{u+1},\ D(x_{u+1}) = (u+1-k)(n-k-u)x_u \quad \textbf{(*).}$$

Thus there exists a SJB of $V(B(n))$.

Easy to see that in any SJB of $V(B(n))$ the subspace spanned by a SJC is an irreducible $\mathrm{sl}(2,\mathbb{C})$-submodule and that (*) applies.

Step 3 S_n action on $V(B(n))$: Orthogonal SJB

Existence of orthogonal SJB follows from:

(a) Existence of some SJB.

(b) U is S_n-linear.

(c) $V(B(n)_k)$ is a multiplicity free S_n-module, for all k (well known).

(d) For a finite group G, a G-invariant inner product on an irreducible G-module is unique upto scalars.

Step 4 Singular Values

Let $J(n)$ be a orthogonal SJB and let $(x_k, \ldots, x_{n-k})$ be a SJC in $J(n)$.

We have from formula (*)

$$\begin{aligned} \langle x_{u+1}, U(x_u)\rangle &= \langle x_{u+1}, x_{u+1}\rangle \\ \langle D(x_{u+1}), x_u\rangle &= (u+1-k)(n-k-u)\langle x_u, x_u\rangle \end{aligned}$$

The left hand sides of the identities above are equal since U and D are adjoints of each other with respect to the standard inner product.

Step 5 Explicit block-diagonalization of $\mathcal{T}_n$

Normalize an orthogonal SJB $J(n)$ to get a orthonormal basis $J'(n)$. The subspace spanned by the vectors in any SJC in $J(n)$ is closed under U and D and thus, by the identity $M^s_{l,k} = \sum_{p=0}^{n} (-1)^{p-s} \binom{p}{s} M^p_{l,p} M^p_{p,k}$ is closed under $\mathcal{T}_n$.

The action of $M^p_{l,p}$ (and hence of $M^s_{l,k}$) on the vectors in $J'(n)$ can be written down explicitly using the singular values.

Since any two SJC's in $J(n)$ from k to $n-k$ look alike, we need to pick just one chain from k to $n-k$, for $k = 0, 1, \ldots, \lfloor n/2 \rfloor$. Thus we get

$m = \lfloor n/2 \rfloor,\ p_k = n - 2k + 1,\ q_k = \binom{n}{k} - \binom{n}{k-1},\ k = 0, \ldots, m.$

Doing this calculation yields: Write

$$\Phi\left(\sum_{r,s,t=0}^{n} x_{r,s}^{t} M_{r,s}^{t}\right) = (N_0, \ldots, N_m),$$

where, for $k = 0, \ldots, \lfloor n/2 \rfloor$, the rows and columns of N_k are indexed by

$\{k, k+1, \ldots, n-k\}$. For $k \leq i, j \leq n-k$ the entry in row i, col j of N_k

is given by **Schrijver's formula**

$$\frac{\sum_{t=0}^{n}\sum_{u=0}^{n}(-1)^{u-t}\binom{u}{t}\binom{n-2k}{u-k}\binom{n-k-u}{i-u}\binom{n-k-u}{j-u}x_{i,j}^{t}}{\sqrt{\binom{n-2k}{i-k}\binom{n-2k}{j-k}}}$$

PART IV

Explicit Construction of SJB:

The linear de Bruijn, Tengbergen, Kruyswijk (BTK) algorithm

Up operator on product of Chains: $M(n; k_1, k_2, \ldots, k_n)$

$n, k_1, \ldots, k_n$ positive integers

$M(n; k_1, \ldots, k_n) = \{(x_1, \ldots, x_n) : 0 \leq x_i \leq k_i, \ \forall i\}$

$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ iff $x_i \leq y_i, \ \forall i$ $r(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$

$r(M(n; k_1, \ldots, k_n)) = k_1 + k_2 + \cdots + k_n$

$\dim\ (V(M(n; k_1, \ldots, k_n))) = (k_1 + 1) \cdots (k_n + 1)$

Two special cases

Uniform case: $M(n, k) = M(n; k, \ldots, k)$, $\dim\ (V(M(n, k))) = (k + 1)^n$

Hypercube: $B(n) = M(n; 1, \ldots, 1)$

Up operator on $M(n; k_1, k_2, \ldots, k_n)$.

Example: $n = 3, k_1 = 3, k_2 = 2, k_3 = 2$

$U((0,2,1)) = (1,2,1) + (0,2,2)$

$U((2,1,1)) = (3,1,1) + (2,2,1) + (2,1,2)$

Two applications, namely, **proof of unimodality of the Schur function specialization** $s_\lambda(1, q, \ldots, q^k)$, $\lambda \vdash n$ (see Exercise 7.75 in **Stanley's** EC-2) and the **product theorem for Peck posets** (see Chapter 6 of the book "Sperner Theory" by **Engel**) *suggest that we consider general chain products and not just hypercubes.*

One approach to constructing an explicit SJB of $V(M(n; k_1, \ldots, k_n))$ is to use tensor products: using (*) define a $\mathrm{sl}(2, \mathbb{C})$ action on $V(M(1; k_i)),\ 0 \leq i \leq n$ by (here we take $k_i + 1 - 0$)

$$U(u) = u + 1,\ D(u+1) = (u+1)(k_i - u)u.$$

and then consider the tensor product representation. Note that, in the general case, U and D are NOT adjoint under the standard inner product. *The starting point of this paper was the observation that a simpler combinatorial approach is to* **linearize** *the famous* **de Bruijn, Tengbergen, and Kruyswijk (1951) (BTK) bijection**.

A **symmetric chain** in a graded rank-n poset P is a sequence $(p_1, \ldots, p_h)$ of elements of P such that p_i covers p_{i-1}, for $i = 2, \ldots h$, and $r(p_1) + r(p_h) = n$, if $h \geq 2$, or else $2r(p_1) = n$, if $h = 1$.

A **symmetric chain decomposition** (SCD) of a graded poset P is a decomposition of P into pairwise disjoint symmetric chains.

de Bruijn, Tengbergen, and Kruyswijk (1951) gave a bijection (essentially, just one figure) that constructs an explicit SCD of $M(2; p, q)$. **This is then inductively used to construct an explicit SCD of $M(n; k_1, k_2, \ldots, k_n)$.**

We can think of SJB's as linear analogs of SCD's. Even more, there is a natural, elementary linear algebraic algorithm, **the linear BTK algorithm**, *that is a linearization of the set theoretic* **BTK bijection**.

MT 1 *The linear BTK algorithm constructs an explicit SJB of $V(M(n, k_1, \ldots, k_n))$. The vectors in this basis have integral coefficients when expressed in the standard basis $M(n, k_1, \ldots, k_n)$.*

The method first constructs an explicit SJB of $V(2; p, q)$ and then uses it repeatedly in the general case.

Can we characterize the basis produced by the linear BTK algorithm? *There is an elegant answer for the hypercube* ($k = 1$).

MT 2 *Let* $O(n)$ *be the SJB produced by the linear BTK algorithm when applied to* $B(n)$.

(i) The elements of $O(n)$ *are orthogonal with respect to* $\langle , \rangle$.

(ii) Let $(x_k, \ldots, x_{n-k})$, $0 \leq k \leq \lfloor n/2 \rfloor$ *be a SJC in* $O(n)$. *Then*

$$\frac{\| x_{u+1} \|}{\| x_u \|} = \sqrt{(u+1-k)(n-k-u)}, \quad k \leq u < n-k.$$

Moreover, this orthogonal SJB is **canonically** defined.

Symmetric Gelfand-Tsetlin (GZ) basis of $V(B(n))$

Consider an irreducible S_n-module V. Since the branching is simple the decomposition of V into irreducible S_{n-1}-modules is canonical. Each of these modules, in turn, decompose canonically into irreducible S_{n-2}-modules. Iterating this construction we get a canonical decomposition of V into irreducible S_1-modules, i.e., one dimensional subspaces. This canonical basis of V, determined upto scalars, is called the **Gelfand-Tsetlin basis** *(GZ-basis).* **Note that it is orthogonal under the (unique upto scalars) S_n-invariant inner product.**

Now observe the following:

(i) If $f : V \to W$ is a S_n-linear isomorphism between irreducibles V, W then the GZ-basis of V goes to the GZ-basis of W.

(ii) Let V be a S_n-module whose decomposition into irreducibles is multiplicity free. By the GZ-basis of V we mean the union of the GZ-bases of the various irreducibles occuring in the (canonical) decomposition of V into irreducibles. **Then the GZ-basis of V is orthogonal wrt any S_n-invariant inner product on V.**

Now consider the S_n action on $V(B(n))$. Since U is S_n-linear, the action is multiplicity free on $V(B(n)_k)$, for all k, and there exists a SJB of $V(B(n))$, it follows from points (i) and (ii) above that there is a **canonically defined** *orthogonal SJB of $V(B(n))$ (upto a common scalar multiple on each symmetric Jordan chain) that consists of the union of the GZ-bases of $V(B(n)_k)$, $0 \leq k \leq n$. We call this basis the* **symmetric Gelfand-Tsetlin basis** *of $V(B(n))$.*

MT 3 *When applied to the poset $B(n)$ the linear BTK algorithm produces the Gelfand-Tsetlin basis of $V(B(n))$.*

Proof is a simple application of **Vershik-Okounkov theory:**

characterization of GZ-basis as simultaneous eigenvectors for the

Young-Jucys-Murphy elements (YJM-elements).

$X_i = (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}S_n, \;\; 1 \leq i \leq n.$

THANK YOU