The geometry of extremal elements in a Lie algebra

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joint work with Cuypers, Ivanyos, in't panhuis, Roozemond, ...

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Outline

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Motivation for extremal elements in Lie algebras

1. Lie algebraic equivalent of long root groups
2. Universal axiom system for point-line spaces of all spherical types
3. Classification of simple Lie algebras

Lie algebra

Definition

A vector space L over a field k supplied with a multipication $[\cdot, \cdot]$ such that

- $[x, x] = 0$ for each $x \in L$
- $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ for each $x, y, z \in L$

- Nonassociative
- Example: $[x, y] = xy - yx$ on an associative algebra
- Linear, first order, version of Lie groups
- Ideals, solvability, simplicity, ...

Extremal elements

Let L be a Lie algebra over k.

Definition

If $\text{char}(k) \neq 2$, then $x \in L$ is *extremal* if $x \neq 0$ and $[x,[x,y]] = 2g_x(y)x$ and (if $\text{char}(k)$ even)

$$[x,[y,[x,z]]] = g_x([y,z])x - g_x(z)[x,y] - g_x(y)[x,z].$$

- Identity: Premet
- Example: rank 1 matrices X with $X^2 = 0$ in $\mathfrak{gl}(k^n)$
- Notation: $E(L)$ or E

Two more examples

Let V be a vector space over k.

- Symplectic form f. Each infinitesimal symplectic transvection, of the form
$$x \mapsto f(x,a)a$$
for $a \in V$, is extremal in $\mathfrak{sp}(V,f)$, the symplectic Lie algebra wrt f.
- Quadratic form κ. Denote by f the associated bilinear form. Each infinitesimal Siegel transvection, of the form
$$x \mapsto f(x,a)b + f(x,b)a$$
for $a, b \in V$ with $\kappa(a) = f(a,b) = \kappa(b) = 0$), is extremal in $\mathfrak{o}(V,\kappa)$, the orthogonal Lie algebra wrt κ.

Automorphisms

Definition

For $x \in L$ and $t \in k$ the map $\exp(x, t) : L \to L$ is given by

$$\exp(x, t)y = y + t[x, y] + t^2 g_x(y)x.$$

Theorem

Then $t \mapsto \exp(x, t)$ *is an injective homomorphism of groups* $k^+ \to \mathrm{Aut}(L)$*.*

Basic properties of extremal elements

Theorem

Let $x, y \in \mathrm{E}$ and $z \in L$.

(i) *The subalgebra $\langle \mathrm{E} \rangle$ of L generated by E is linearly spanned by E.*

(ii) *On $\langle \mathrm{E} \rangle$ there is a unique symmetric bilinear form g with value $g_x(y)$ at (x, y).*

(iii) *For all $a, b, c \in \langle \mathrm{E} \rangle$, we have $g([a, b], c) = g(a, [b, c])$. In particular, the form g of (ii) is associative.*

Degenerate extremal elements

Definition

$x \in E(L)$ is a *sandwich* if $g_x = 0$.

- Equivalently: $\mathrm{ad}_x^2 = \mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x = 0$ for all $y \in L$
- Sandwiches lie in $\mathrm{Rad}(g) = \{x \in L \mid g(x, L) = 0\}$

Nilpotent ideals

Proposition

Assume $L = \langle \mathrm{E} \rangle$. Then $\mathrm{Rad}(g)$ is a nilpotent ideal of L containing all sandwiches of L.

- If $x \in \mathrm{E}(L)$ is not a sandwich, then $x + \mathrm{Rad}(g) \in E(L/\mathrm{Rad}(g))$ is not a sandwich.
- In passing from L to $L/\mathrm{Rad}(g)$, all sandwiches disappear and all non-sandwich extremals remain.

Algebraic groups

G: k-split connected reductive linear algebraic group with reductive rank n and semisimple rank ℓ

T: a maximal split torus of G

Φ: the root system with respect to T

X: the character group of T

Y: its dual so there is a bilinear pairing $\langle \cdot, \cdot \rangle \to \mathbb{Z}$

Basis $e_1, \ldots, e_n$ of X and dual $f_1, \ldots, f_n$ of Y

$\Phi^\star$: coroot system in Y with a bijective correspondence such that $\langle \alpha, \alpha^\star \rangle = 2$

L: Lie algebra of G

H: Lie subalgebra of T

Chevalley basis of the Lie algebra of an algebraic group

Theorem

L has basis elements e_α for $\alpha \in \Phi$ and $h_i \in H$ for $i = 1, \ldots, n$ with structure constants:

$$\begin{aligned}
[h_i, h_j] &= 0 \\
[e_\alpha, h_i] &= \langle \alpha, f_i \rangle e_\alpha, \\
[e_{-\alpha}, e_\alpha] &= \sum_{i=1}^{n} \langle e_i, \alpha^\star \rangle h_i, \\
[e_\alpha, e_\beta] &= \begin{cases} N_{\alpha\beta} e_{\alpha+\beta} & \text{for } \alpha + \beta \in \Phi, \\ 0 & \text{for } \alpha + \beta \notin \Phi,\ \beta \neq -\alpha, \end{cases}
\end{aligned}$$

for certain integral constants $N_{\alpha,\beta}$.

Extremal elements in Chevalley basis

- If $\alpha \in \Phi$ long root, then $e_\alpha \in \mathrm{E}(L)$.
- Idea: $[e_\alpha, [e_\alpha, e_\beta]] \subseteq ke_{2\alpha+\beta}$
- $G \to \mathrm{Aut}(L)$ via adjoint representation
- The full G-orbit on e_α is contained in $\mathrm{E}(L)$.

Previous examples

- $\mathrm{E}(\mathfrak{sl}(V))$: Dynkin type A_{n-1} if $\dim(V) = n$.
- $\mathrm{E}(\mathfrak{sp}(V, f))$: Dynkin type C_n if $\dim(V) = 2n$ and f non-degenerate.
- $\mathrm{E}(\mathfrak{o}(V, \kappa))$: Dynkin type D_n if $\dim(V) = 2n$ and type B_n if $\dim(V) = 2n + 1$ if κ non-degenerate.

Fundamental properties of extremal elements

There are five essentially different positions two extremal elements can be in with respect to each other.

Definition

For $i \in \{-2, -1, 0, 1, 2\}$, define relations E_i on E as follows.

i	$x\mathrm{E}_i y$	name
-2	$kx = ky$	identical
-1	$[x, y] = 0,\ kx + ky \subseteq \mathrm{E} \cup \{0\},\ kx \neq ky$	strongly commuting
0	$[x, y] = 0,\ kx + ky \not\subseteq \mathrm{E} \cup \{0\}$	polar
1	$[x, y] \neq 0,\ g(x, y) = 0$	special
2	$g(x, y) \neq 0$	hyperbolic

Two or generator cases

Lemma

Let $x, y, z \in \mathrm{E}$.

(i) *The subalgebra of L generated by x and y is at most 3-dimensional. It is commutative if $x\mathrm{E}_i y$ with $i \leq 0$, it is isomorphic to the Heisenberg algebra if $x\mathrm{E}_1 y$, and it is isomorphic to $\mathfrak{sl}(k^2)$ if $x\mathrm{E}_2 y$.*

(ii) *The subalgebra $\langle x, y, z \rangle$ is at most 8-dimensional. It is nilpotent, isomorphic to (a possibly twisted form of) $\mathfrak{sl}(k^3)$ (like $\mathfrak{u}(\mathbb{R}^3, \kappa)$ where κ is a non-compact unitary form), or an extension of $\mathfrak{sl}(k^2)$ by a nilpotent ideal.*

Five generators?

- $\dim(L) < \infty$ if L is generated by a finite number of extremal elements is finite.
- There is a nilpotent Lie algebra of maximal dimension $f(N)$ for any given finite number N of extremal generators.
- $f(2) = 3$ also attained by $\mathfrak{sl}(k^2)$
  $f(3) = 8$ also attained by $\mathfrak{sl}(k^3)$
  $f(4) = 28$ also attained by $\mathfrak{o}(k^8, \kappa)$
- $f(5) = 537$. The Lie algebras of types E_6, E_7, and E_8 are generated by 5 extremal elements.

Problem

Is there a ‘generic’ Lie algebra of maximal possible dimension 537 in the 5 generator case?

‘genericity’ is made precise by Draisma and in’t panhuis.

Outline

Classification in three steps

Theorem

If E contains a non-sandwich, L is simple, and $k = \overline{k}$, then L is the Lie algebra of an algebraic group.

Due to Benkart, Block, et al. Easy by use of root group elements $\exp(x, t)$ $(x \in \mathrm{E}, t \in k)$. New —geometric— approach to classification:

- First, produce a point-line geometry from L.
- Second, characterize these geometries as certain (root) shadow spaces of spherical buildings.
- Third, given a root shadow space $(\mathcal{E}, \mathcal{F})$, show that, up to isomorphism, there is at most one simple Lie algebra whose geometry is isomorphic to $(\mathcal{E}, \mathcal{F})$. incomplete.

Existence of extremal elements

Theorem

If k is algebraically closed of characteristic distinct from 2 and 3 and L is finite-dimensional and simple, then L has an extremal element.

- result by Premet
- self-contained proof by Tange
- continue in this vein...replacing intricate long literature with slick proofs using geometry

The Witt example

- $\mathrm{char}(k) = 5$
- $W_{1,1}(5)$: vector space over k with basis $z^i\partial_z$, for $i = 0, \ldots, 4$.
- Lie bracket on $W_{1,1}(5)$ determined by

$$[z^i\partial_z, z^j\partial_z] \;:=\; (j-i)z^{i+j-1}\partial_z,$$

with the convention that $z^i = 0$ whenever $i \notin \{0, \ldots, 4\}$.
- $x = -z^2\partial_z$ is extremal in $W_{1,1}(5)$ and not a sandwich.
- $z^3\partial_z$ is a sandwich in $W_{1,1}(5)$.
- $W_{1,1}(5)$ is not generated by $\mathrm{E}(W_{1,1}(5))$.

Existence of more extremal elements

Theorem

char$(k) \neq 2, 3$ *and* L *simple. Suppose that* L *contains an extremal element that is not a sandwich. Then either* k *has characteristic* 5 *and* $L \cong W_{1,1}(5)$ *or* $L = \langle E \rangle$.

- result joint with Ivanyos and Roozemond
- Jacobson-Morozov to produce second extremal element
- next use a filtration as to be discussed
- 12 occurs in denominators of formulas used

Filtration wrt an extremal element

Proposition

$x, y \in E$ with $g_x(y) = 1$. Write
$U = \{u \in L \mid g_x(u) = g_y(u) = g_x([y,u]) = 0\}$.

1. *There is a $\mathbb{Z}$-grading $L = \bigoplus_i L_i(x,y)$ with $L_{-2}(x,y) = kx$, $L_2(x,y) = ky$, $L_0(x,y) = N_L(kx) \cap N_L(ky)$, $L_{-1}(x,y) = [x,U]$, and $L_1(x,y) = [y,U]$.*
2. *There is a filtration*

$$L_{\leq -2}(x) \subseteq L_{\leq -1}(x) \subseteq L_{\leq 0}(x) \subseteq L_{\leq 1}(x) \subseteq L_{\leq 2}(x) = L,$$

where $L_{\leq i}(x) = \sum_{j=-2}^{i} L_j(x,y)$. Moreover, $L_{\leq 1}(x) = \{z \in L \mid g_x(z) = 0\}$, $L_{\leq 0}(x) = N_L(kx)$, and $L_{\leq -1}(x) = kx + [x, L_{\leq 1}(x)]$.

So $L_{\leq i}(x)$ are independent of the choice of y.

Extremal points in projective space

- E_i on E lead to projective relations $\mathcal{E}_i$ on $\mathcal{E}$, the projective points of members of E.

The geometry of $L = \langle E \rangle$

- The relations E_i are symmetric and partition $\mathrm{E} \times \mathrm{E}$, and the $\mathcal{E}_i$ partition $\mathcal{E} \times \mathcal{E}$.
- The geometry of L:
  - point set $\mathcal{E}$;
  - line set $\mathcal{F}$: all projective lines of L contained in $\mathcal{E}$.

Fundamental properties of extremal elements

Theorem

$L = \langle \mathrm{E}(L) \rangle$ has no sandwiches. The space $(\mathcal{E}, \mathcal{F})$ with the symmetric relations $\mathcal{E}_i$ ($i \in \{-2, -1, 0, 1, 2\}$) on $\mathcal{E}$ satisfies the following properties, where we write $\mathcal{E}_{\leq i}$ for $\cup_{j \leq i} \mathcal{E}_j$.

(A) *The relation $\mathcal{E}_{-2}$ is equality on $\mathcal{E}$.*

(B) *The relation $\mathcal{E}_{-1}$ is collinearity of distinct points of $\mathcal{E}$.*

(C) *There is a map $\mathcal{E}_1 \to \mathcal{E}$, denoted by $(u, v) \mapsto [u, v]$ such that, if $(u, v) \in \mathcal{E}_1$ and $x \in \mathcal{E}_i(u) \cap \mathcal{E}_j(v)$, then $[u, v] \in \mathcal{E}_{\leq i+j}(x)$.*

(D) *For each $(x, y) \in \mathcal{E}_2$, we have $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$.*

(E) *For each $x \in \mathcal{E}$, the subsets $\mathcal{E}_{\leq -1}(x)$ and $\mathcal{E}_{\leq 0}(x)$ are subspaces of $(\mathcal{E}, \mathcal{F})$.*

(F) *For each $x \in \mathcal{E}$, the subset $\mathcal{E}_{\leq 1}(x)$ is a hyperplane of $(\mathcal{E}, \mathcal{F})$.*

Outline

Root filtration spaces

Definition

Let $(\mathcal{E}, \mathcal{F})$ be a partial linear space. For $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$, a quintuple of disjoint symmetric relations partitioning $\mathcal{E} \times \mathcal{E}$, we call $(\mathcal{E}, \mathcal{F})$ a *root filtration space with filtration* $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ if the properties (A)–(F) are satisfied.

Examples of root filtration spaces

- geometry of a Lie algebra generated by extremal elements
- polar spaces: $\mathcal{E}_{-1} = \mathcal{E}_1 = \emptyset$
- generalized hexagons: $\mathcal{E}_0 = \emptyset$
- root shadow spaces of spherical buildings
- disjoint commuting unions
- $\mathcal{E}(\mathcal{M}, \mathcal{N})$ for a non-degenerate pair $(\mathcal{M}, \mathcal{N})$ of dual projective spaces

Non-degenerate root filtration spaces

Definition

A root filtration space $(\mathcal{E}, \mathcal{F})$ is called *non-degenerate* if it satisfies:

(G) For each $x \in \mathcal{E}$ the set $\mathcal{E}_2(x)$ is not empty.

(H) The graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.

- polar spaces are degenerate

Abstract root subgroups

- If $L = \langle E \rangle$ has no sandwiches, then $\{\{\exp(x,t) \mid t \in k\} \mid x \in E(L)\}$ is a set of abstract root groups as defined by Timmesfeld.
- If $\mathcal{E}$ is a set of abstract root groups, then there is a natural $\mathcal{F}$ such that $(\mathcal{E}, \mathcal{F})$ is a root filtration space.
- Using $\overline{k} = k$ can identify L with Lie algebra of an algebraic group.

Important properties of root filtration spaces

Lemma

In a root filtration space $(\mathcal{E}, \mathcal{F})$ the following properties hold.

(i) *For each $i \in \{-2, \ldots, 2\}$ and each $x \in \mathcal{E}$, the subset $\mathcal{E}_{\leq i}(x)$ is a subspace of $(\mathcal{E}, \mathcal{F})$.*

(ii) *If $(u, v) \in \mathcal{E}_1$, then $[u, v]$ is the unique common neighbor of both u and v in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of $(\mathcal{E}, \mathcal{F})$.*

(iii) *If $(u, v) \in \mathcal{E}_1$, then $\mathcal{E}_0(u) \cap \mathcal{E}_2(v) \subseteq \mathcal{E}_1([u, v])$.*

(iv) *If $(x, y) \in \mathcal{E}_0$ and $z \in \mathcal{E}_{-1}(y)$, then either $z \in \mathcal{E}_{\leq 0}(x)$, or $z \in \mathcal{E}_1(x)$ and $\mathcal{E}_{-1}(x, y, z) = \{[x, z]\}$.*

(v) *If (x, q) and (u, z) belong to $\mathcal{E}_1$ whereas $u = [x, q]$ and $q = [u, z]$, then $(x, z) \in \mathcal{E}_2$.*

(vi) *If P is a pentagon in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ (that is, the induced subgraph is a pentagon), then each distinct non-collinear pair of points of P is polar.*

(vii) *If $(u, v) \in \mathcal{E}_1$, then $\mathcal{E}_{-1}(u) \cap \mathcal{E}_0([u, v]) \subseteq \mathcal{E}_1(v)$.*

Classification of non-denegerate root filtration spaces

Theorem

Let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space.

(i) *If some line in $\mathcal{F}$ is contained in a unique maximal singular subspace, then one of*

I. *The rank of a maximal singular subspace is 1 and $(\mathcal{E}, \mathcal{F})$ is a root shadow space of type G_2.*

II. *The rank of a maximal singular subspace is at least 2, there is a point that belongs to at least 3 maximal singular subspaces, and $(\mathcal{E}, \mathcal{F})$ is a root shadow space of type $(\mathrm{B}|\mathrm{C})_{3,2}$.*

III. *The rank of a maximal singular subspace is at least 2, on each point there are precisely 2 maximal singular subspaces, and $(\mathcal{E}, \mathcal{F})$ is isomorphic to $\mathcal{E}(\mathcal{M}, \mathcal{N})$ for a certain pair $(\mathcal{M}, \mathcal{N})$ of projective spaces in duality.*

(ii) *If some line in $\mathcal{F}$ is contained in more than one maximal singular subspace, then $\mathcal{E}_1 \neq \emptyset$ and $(\mathcal{E}, \mathcal{F})$ is the root shadow space of a building of type B_n, C_n $(n \geq 4)$, D_n $(n \geq 4)$, E_6, E_7, E_8, or F_4.*

About the proof of the non-denegerate case

- Cohen-Ivanyos show that if $(\mathcal{E}, \mathcal{F})$ does not satisfy (i), it satisfies the conditions of Kasikova-Shult.
- Kasikova-Shult use results of Cohen-Cooperstein to prove that $(\mathcal{E}, \mathcal{F})$ is the shadow space of a building.
- Cohen-Cooperstein produce all shadows of other types than points and lines of the building to be reconstructed and apply Tits' local approach to spherical buildings to recognize $(\mathcal{E}, \mathcal{F})$ as the root shadow space of that building.

Classification of Lie algebras with denegerate root filtration spaces

Theorem

Let L be a Lie algebra over a field k of size at least 3 generated by its extremal elements. Assume

(i) *the sets $\mathcal{E}_{-1}$ and $\mathcal{E}_1$ are empty;*

(ii) *the graph $(\mathcal{E}, \mathcal{E}_2)$ is connected;*

(iii) *the set $\mathcal{E}_0$ is nonempty.*

Then $(\mathcal{E}, \mathcal{E}_0)$ is the collinearity graph of a non-degenerate polar space.

About the proof of the degenerate case

- use the hyperbolic lines from the Lie algebra
- apply theorem by Cuypers

Transversal cocliques in dual affine planes

A *dual affine plane* is a point-line space obtained from a projective plane by removing a point and all lines containing that point.

A *transversal coclique* in a dual affine plane is the set of points forming such a removed line together with the removed point.

Cuypers' theorem

Theorem

Let $(\mathcal{E}, \mathcal{H})$ be a point-line space. Let $\sim$ denote collinearity for distinct points and $\perp$ its complement. Suppose $(\mathcal{E}, \mathcal{H})$ satisfies the following six conditions.

(i) *$(\mathcal{E}, \mathcal{H})$ is a connected partial linear space that is not linear.*

(ii) *Each line contains at least four points.*

(iii) *The subspace of $(\mathcal{E}, \mathcal{H})$ generated by any triple of points x, y, z with $x \sim y \sim z \perp x$ is a dual affine plane.*

(iv) *If $p \perp$ with 2 points of a transversal coclique T, then $p \perp T$.*

(v) *if x, y are points with $x^{\perp} \subseteq y^{\perp}$, then $x = y$.*

(vi) *$(\mathcal{E}, \perp)$ is connected.*

Then $(\mathcal{E}, \perp)$ is collinearity graph of a non-degen. polar space.

Outline

Hyperbolic lines in the degenerate case

For arbitrary degenerate root filtration spaces, hyperbolic lines are the additional structure needed.

Cuypers' theorem needs lines of size at least 4.

For line size 3, the Fischer spaces turn up.

Fischer spaces

Definition

A *Fischer space* is a partial linear space in which each plane is isomorphic to a dual affine plane of order two or to an affine plane of order three.

- So lines have size 3.
- Buekenhout's geometric interpretation of Fischer's work on 3-transpositions.

Fischer spaces from Lie algebras

Theorem

Let $L = \langle E \rangle$ and $k = \mathbb{Z}/2\mathbb{Z}$ such that

(i) *the sets $\mathcal{E}_{-1}$ and $\mathcal{E}_1$ are empty;*

(ii) *the graph $(\mathcal{E}, \mathcal{E}_2)$ is connected;*

(iii) *the set $\mathcal{E}_0$ is nonempty.*

Then $(\mathcal{E}, \mathcal{L})$, where $\mathcal{L}$ is the collection of hyperbolic lines of L, is a connected Fischer space.

- hyperbolic line $= \mathcal{E} \cap \langle x, y \rangle$ for (x, y) hyperbolic pair.
- Which Fischer spaces occur?
- To be answered by reverse construction?

Lie algebras from Fischer spaces

$(\mathcal{E}, \mathcal{H})$: a Fischer space.

A: the vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\mathcal{E}$.

$*$: multiplication on A determined by

$$x * y = \begin{cases} x + y + z & \text{if } \{x, y, z\} \in \mathcal{H} \\ 0 & \text{otherwise} \end{cases}$$

f: form on A determined by $f(x, y) = 1$ if $x \sim y$ and $f(x, y) = 0$ otherwise, for $x, y \in \mathcal{E}$.

Lemma

The symplectic form f is associative: $f(x, y * z) = f(x * y, z)$ for all $x, y, z \in A$, so $\mathrm{Rad}(f)$ is an ideal of A.

The Lie algebra of a Fischer space

work by Cuypers, Gramlich, Horn, in't panhuis, Shpectorov

- Now the quotient algebra $A/\mathrm{Rad}(f)$ is a Lie algebra if and only if each affine plane of $(\mathcal{E}, \mathcal{H})$ belongs to $\mathrm{Rad}(f)$.
- This Lie algebra $A/\mathrm{Rad}(L)$ may collapse, e.g., for F_{24}.
- F_{22} leads to $A/\mathrm{Rad}(f) \cong \mathrm{Lie}({}^2\mathrm{E}_6(2))$, giving a geometric proof of a group embedding.

Outline

Third step in the characterization of classical Lie algebras by means of root filtration spaces has not yet been finalized:

Problem

Under which conditions (e.g. nondegeneracy) is it true that, for a given root filtration space $(\mathcal{E}, \mathcal{F})$, there is at most one simple Lie algebra up to isomorphism whose root filtration space is isomorphic to $(\mathcal{E}, \mathcal{F})$?

Need embeddability of the geometry in a projective space.

Which root filtration spaces originate from a Lie algebra?

Problem

Which root filtration spaces originate from a Lie algebra?

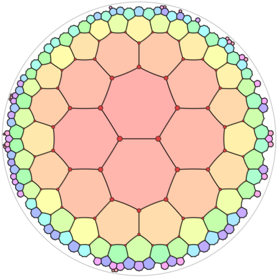
- If f is a non-degenerate symplectic form on V, then the corresponding polar space originates from a Lie algebra.
- If $\dim(V) > 3$ and κ is a non-degenerate quadratic form on V, then the corresponding polar space does not originate from a Lie algebra.

Do all extremal elements lie on a quadric?

Problem

Suppose $\mathrm{char}(k) = 2$ and $L = \langle \mathrm{E} \rangle$. Is there a non-zero quadratic form κ such that $\mathcal{E}$ is contained in the quadric defined by κ?

Nanri — Thank you



R14.3' {7, 3}: order 2184, 156 septagons, 546 edges, 364 vertices