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Wold decomposition for doubly commuting isometries

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WOLD DECOMPOSITION FOR DOUBLY COMMUTING ISOMETRIES

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Abstract. In this paper, we obtain a complete description of the class of n-tuples ($n \geq 2$) of doubly commuting isometries. In particular, we present a several variables analogue of the Wold decomposition for isometries on Hilbert spaces. Our main result is a generalization of M. Slocinski's Wold-type decomposition of a pair of doubly commuting isometries.

1. Introduction

Let V be an isometry on a Hilbert space $\mathcal{H}$, that is, $V^*V = I_{\mathcal{H}}$. A closed subspace $\mathcal{W} \subseteq \mathcal{H}$ is said to be *wandering subspace* for V if $V^k\mathcal{W} \perp V^l\mathcal{W}$ for all $k, l \in \mathbb{N}$ with $k \neq l$, or equivalently, if $V^m\mathcal{W} \perp \mathcal{W}$ for all $m \geq 1$. An isometry V on $\mathcal{H}$ is said to be a *unilateral shift* or *shift* if

$$\mathcal{H} = \sum_{m\geq 0} \oplus V^m\mathcal{W},$$

for some wandering subspace $\mathcal{W}$ for V.

For a shift V on $\mathcal{H}$ with a wandering subspace $\mathcal{W}$ we have

$$\mathcal{H} \ominus V\mathcal{H} = \sum_{m\geq 0} \oplus V^m\mathcal{W} \ominus V(\sum_{m\geq 0} \oplus V^m\mathcal{W}) = \sum_{m\geq 0} \oplus V^m\mathcal{W} \ominus \sum_{m\geq 1} \oplus V^m\mathcal{W} = \mathcal{W}.$$

In other words, the wandering subspace of a shift is unique and is given by $\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}$. The dimension of the wandering subspace of a shift is called the *multiplicity* of the shift.

One of the most important results in the study of operator algebras, operator theory and stochastic process is the Wold decomposition theorem ([11], see also page 3 in [6]), which states that every isometry on a Hilbert space is either a shift, or a unitary, or a direct sum of shift and unitary (see Theorem 2.1). More precisely, the Wold decomposition, or results analogous of the Wold decomposition plays an important role in many areas of operator algebras and operator theory, including the invariant subspace problems for Hilbert function spaces (cf. [5], [8]).

The natural question then becomes: Let $n \geq 2$ and $V = (V_1, \ldots, V_n)$ be an n-tuple of commuting isometries. Does there exists a Wold-type decomposition of V? What happens if we have a family of isometries?

Several interesting results have been obtained in many directions. For instance, in [10] Suciu developed a structure theory for semigroup of isometries (see also [2], [3], [4]). However, to get a more precise result it will presumably always be necessary to make additional assumptions on

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the family of operators. Before proceeding, we recall the definition of the doubly commuting isometries.

Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of commuting isometries on $\mathcal{H}$. Then V is said to *doubly commute* if

$$V_i V_j^* = V_j^* V_i,$$

for all $1 \leq i < j \leq n$.

The simplest example of an n-tuple of doubly commuting isometries is the tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ by the coordinate functions on the Hardy space $H^2(\mathbb{D}^n)$ over the polydisc $\mathbb{D}^n$ ($n \geq 2$).

In [9], M. Slocinski obtained an analogous result of Wold decomposition theorem for a pair of doubly commuting isometries.

Theorem 1.1. *(M. Slocinski) Let $V = (V_1, V_2)$ be a pair of doubly commuting isometries on a Hilbert space $\mathcal{H}$. Then there exists a unique decomposition*

$$\mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu},$$

where $\mathcal{H}_{ij}$ are joint V-reducing subspace of $\mathcal{H}$ for all $i, j = s, u$. Moreover, V_1 on $\mathcal{H}_{i,j}$ is a shift if $i = s$ and unitary if $i = u$ and that V_2 is a shift if $j = s$ and unitary if $j = u$.

We refer to [4] for a new proof of Slocinski's result.

Slocinski's Wold-type decomposition does not hold for general tuples of commuting isometries (cf. Example 1 in [9]). However, if V is an n-tuple of commuting isometries and that

$$\dim \ker \left(\prod_{i=1}^{n} V_i^* \right) < \infty,$$

then V admits a Wold-type decomposition (see Theorem 2.4 in [1]).

In this paper, we obtain a Wold-type decomposition for tuples of doubly commuting isometries. We extend the ideas of M. Slocinski on the Wold-type decomposition for a pair of isometries to the multivariable case ($n \geq 2$). Our approach is simple and based on the classical Wold decomposition for a single isometry. Moreover, our method yields a new proof of Slocinski's result for the base case $n = 2$.

The paper is organized as follows. In Section 2, we set up notations and definitions and establish some preliminary results. In Section 3, we prove our main result and some of its consequences.

2. Preparatory results

In this section we recall the Wold decomposition theorem with a new proof and present some elementary facts concerning doubly commuting isometries.

We begin with the Wold decomposition of an isometry.

Theorem 2.1. *(H. Wold) Let V be an isometry on $\mathcal{H}$. Then $\mathcal{H}$ admits a unique decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$, where $\mathcal{H}_s$ and $\mathcal{H}_u$ are V-reducing subspaces of $\mathcal{H}$ and $V|_{\mathcal{H}_s}$ is a shift and $V|_{\mathcal{H}_u}$*

is unitary. Moreover,

$$\mathcal{H}_s = \overset{\infty}{\underset{m=0}{\oplus}} V^m \mathcal{W} \qquad \textit{and} \qquad \mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m \mathcal{H},$$

where $\mathcal{W} = ran(I - VV^*)$ *is the wandering subspace for* V*.*

Proof. Let $\mathcal{W} = \operatorname{ran}(I - VV^*)$ be the wandering subspace for V and $\mathcal{H}_s = \oplus_{m=0}^{\infty} V^m \mathcal{W}$. Consequently, $\mathcal{H}_s$ is a V-reducing subspace of $\mathcal{H}$ and that $V|_{\mathcal{H}_s}$ is an isometry. On the other hand, for all $l \geq 0$,

$$\begin{aligned}(V^l \mathcal{W})^{\perp} &= (V^l \operatorname{ran}(I - VV^*))^{\perp} = \operatorname{ran}(I - V^l(I - VV^*)V^{*l}) \\ &= \operatorname{ran}[(I - V^l V^{*l}) \oplus V^{l+1} V^{*\,l+1}] = \operatorname{ran}(I - V^l V^{*l}) \oplus \operatorname{ran} V^{l+1} \\ &= (V^l \mathcal{H})^{\perp} \oplus V^{l+1} \mathcal{H}.\end{aligned}$$

Consequently, we have

$$\mathcal{H}_u := \mathcal{H}_s^{\perp} = \bigcap_{l=0}^{\infty} V^l \mathcal{H}.$$

Uniqueness of the decomposition readily follows from the uniqueness of the wandering subspace $\mathcal{W}$ for V. This completes the proof. ■

We now introduce some notation which will remain fixed for the rest of the paper. Given an integer $1 \leq m \leq n$, we denote the set $\{1, \ldots, m\}$ by I_m. In particular, $I_n = \{1, \ldots, n\}$.

Define $\mathcal{W}_i := \operatorname{ran}(I - V_i V_i^*)$ for each $1 \leq i \leq n$ and

$$\mathcal{W}_A := \operatorname{ran}(\prod_{i \in A} (I - V_i V_i^*)),$$

where A is a non-empty subset of I_m and $1 \leq m \leq n$.

By doubly commutativity of V it follows that

$$(I - V_i V_i^*)(I - V_j V_j^*) = (I - V_j V_j^*)(I - V_i V_i^*),$$

for all $i \neq j$. In particular,

$$\{(I - V_i V_i^*)\}_{i=1}^n$$

is a family of commuting orthogonal projections on $\mathcal{H}$. Therefore, for all non-empty subset A of I_m $(1 \leq m \leq n)$ we have

$$\mathcal{W}_A = \operatorname{ran}(\prod_{i \in A} (I - V_i V_i^*)) = \bigcap_{i \in A} \operatorname{ran}(I - V_i V_i^*)) = \bigcap_{i \in A} \mathcal{W}_i. \tag{2.1}$$

The following simple result plays a basic role in describing the class of n-tuples of doubly commuting isometries.

Proposition 2.2. *Let* $V = (V_1, \ldots, V_n)$ *be an* n*-tuple* $(n \geq 2)$ *of doubly commuting isometries on* $\mathcal{H}$ *and* A *be a non-empty subset of* I_m *for* $1 \leq m \leq n$*. Then* $\mathcal{W}_A$ *is a* V_j*-reducing subspace of* $\mathcal{H}$ *for all* $j \in I_n \setminus A$*.*

Proof. By doubly commutativity of V we have

$$V_j(I - V_iV_i^*) = (I - V_iV_i^*)V_j,$$

for all $i \neq j$, and thus

$$V_j(\prod_{i\in A}(I - V_iV_i^*)) = (\prod_{i\in A}(I - V_iV_i^*))V_j,$$

for all $j \in I_n \setminus A$, that is,

$$V_jP_{\mathcal{W}_A} = P_{\mathcal{W}_A}V_j,$$

where $P_{\mathcal{W}_A}$ is the orthogonal projection of $\mathcal{W}_A$ onto $\mathcal{H}$. This completes the proof. ■

To complete this section we will use the preceding proposition to obtain the generalized wandering subspaces for n-tuple of doubly commuting isometries.

Corollary 2.3. *Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting isometries on $\mathcal{H}$ and $m \leq n$. Then for each non-empty subset A of I_m and $j \in I_n \setminus A$,*

$$\mathcal{W}_A \ominus V_j\mathcal{W}_A = ran(\prod_{i\in A}(I - V_iV_i^*)(I - V_jV_j^*)) = \Big(\bigcap_{i\in A}\mathcal{W}_i\Big) \bigcap \mathcal{W}_j.$$

Proof. Doubly commutativity of V implies that

$$\prod_{i\in A}(I - V_iV_i^*)(I - V_jV_j^*) = \prod_{i\in A}(I - V_iV_i^*) - V_j(\prod_{i\in A}(I - V_iV_i^*))V_j^*.$$

By Proposition 2.2 we have $V_j\mathcal{W}_A \subseteq \mathcal{W}_A$ for all $j \notin A$. Moreover

$$V_j\mathcal{W}_A = \text{ran}[V_j\prod_{i\in A}(I - V_iV_i^*)V_j^*],$$

and hence

$$\mathcal{W}_A \ominus V_j\mathcal{W}_A = \text{ran}(\prod_{i\in A}(I - V_iV_i^*) - V_j(\prod_{i\in A}(I - V_iV_i^*))V_j^*) = \text{ran}(\prod_{i\in A}(I - V_iV_i^*)(I - V_jV_j^*)),$$

for all $j \notin A$. The second equality follows from (2.1). This completes the proof. ■

3. The Main Theorem

In this section we will prove the main result of this paper.

Theorem 3.1. *Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting isometries on $\mathcal{H}$. Then there exists 2^n orthogonal joint V-reducing subspaces $\{\mathcal{H}_A : A \subseteq I_n\}$ (counting the trivial subspace $\{0\}$) such that*

$$\mathcal{H} = \sum_{A\subseteq I_n} \oplus \mathcal{H}_A,$$

and for each $A \subseteq I_n$ and $\mathcal{H}_A \neq \{0\}$, $V_i|_{\mathcal{H}_A}$ is a shift if $i \in A$ and unitary if $i \in I_n \setminus A$ for all $i = 1, \ldots, n$. Moreover, the above decomposition is unique, in the sense that

$$\mathcal{H}_A = \sum_{k_i\in\mathbb{N}} \oplus(\prod_{i\in A}V_i^{k_i})(\bigcap_{k_j\in\mathbb{N}}(\prod_{j\in I_n\setminus A}V_j^{k_j})\mathcal{W}_A),$$

for all $A \subseteq I_n$.

Proof. We prove the following more general statement: given any $m \in \{2, \ldots, n\}$ there exists 2^m joint $(V_1, \ldots, V_m)$-reducing subspaces $\{\mathcal{H}_A : A \subseteq I_m\}$ such that

$$\mathcal{H} = \sum_{A \subseteq I_m} \oplus \mathcal{H}_A,$$

where for each $A \subseteq I_m$,

$$\mathcal{H}_A = \sum_{k_i \in \mathbb{N}} \oplus (\prod_{i \in A} V_i^{k_i})(\bigcap_{k_j \in \mathbb{N}} (\prod_{j \in I_m \setminus A} V_j^{k_j})\mathcal{W}_A).$$

We shall prove the above statement using mathematical induction.
For $m = 2$: By applying the Wold decomposition theorem, Theorem 2.1, to the isometry V_1 on $\mathcal{H}$ we have

$$\mathcal{H} = \sum_{k_1 \in \mathbb{N}} \oplus V_1^{k_1} \mathcal{W}_1 \oplus \bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H}.$$

As $\mathcal{W}_1$ is a V_2-reducing subspace, it follows from the Wold decomposition theorem for the isometry $V_2|_{\mathcal{W}_1} \in \mathcal{L}(\mathcal{W}_1)$ that

$$\mathcal{W}_1 = \sum_{k_2 \in \mathbb{N}} \oplus V_2^{k_2}(\mathcal{W}_1 \ominus V_2 \mathcal{W}_1) \oplus \bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 = \sum_{k_2 \in \mathbb{N}} \oplus V_2^{k_2}(\mathcal{W}_1 \bigcap \mathcal{W}_2) \oplus \bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1,$$

where the second equality follows from Corollary 2.3. Consequently,

$$\begin{aligned}
\mathcal{H} &= \sum_{k_1 \in \mathbb{N}} \oplus V_1^{k_1} \mathcal{W}_1 \oplus \bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H} \\
&= \sum_{k_1 \in \mathbb{N}} \oplus V_1^{k_1} \Big(\sum_{k_2 \in \mathbb{N}} \oplus V_2^{k_2} \Big(\mathcal{W}_1 \bigcap \mathcal{W}_2 \Big) \oplus \bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big) \oplus \bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H} \\
&= \sum_{k_1, k_2 \in \mathbb{N}} \oplus V_1^{k_1} V_2^{k_2} \Big(\mathcal{W}_1 \bigcap \mathcal{W}_2 \Big) \oplus \sum_{k_1 \in \mathbb{N}} \oplus V_1^{k_1} \Big(\bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big) \oplus \bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H}.
\end{aligned}$$

Furthermore, the Wold decomposition of the isometry V_2 on $\mathcal{H}$

$$\mathcal{H} = \sum_{k_2 \in \mathbb{N}} \oplus V_2^{k_2} \mathcal{W}_2 \oplus \bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{H},$$

yields

$$V_1^{k_1} \mathcal{H} = \sum_{k_2 \in \mathbb{N}} \oplus V_2^{k_2} V_1^{k_1} \mathcal{W}_2 \oplus \bigcap_{k_2 \in \mathbb{N}} V_1^{k_1} V_2^{k_2} \mathcal{H},$$

for all $k_1 \in \mathbb{N}$. From this we infer that

$$\bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H} = \sum_{k_2 \in \mathbb{N}} \oplus V_2^{k_2} \Big(\bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{W}_2 \Big) \oplus \bigcap_{k_1, k_2 \in \mathbb{N}} V_1^{k_1} V_2^{k_2} \mathcal{H}.$$

Therefore,
(3.1)

$$\mathcal{H} = \sum_{\boldsymbol{k} \in \mathbb{N}^2} \oplus V^{\boldsymbol{k}} \Big(\mathcal{W}_1 \bigcap \mathcal{W}_2 \Big) \oplus \sum_{k_1 \in \mathbb{N}} \oplus V_1^{k_1} \Big(\bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big) \oplus \sum_{k_2 \in \mathbb{N}} \oplus V_2^{k_2} \Big(\bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{W}_2 \Big) \oplus \bigcap_{\boldsymbol{k} \in \mathbb{N}^2} V^{\boldsymbol{k}} \mathcal{H},$$

that is,

$$\mathcal{H} = \sum_{A \subseteq I_2} \oplus \mathcal{H}_A,$$

where $V^{\boldsymbol{k}} = V_1^{k_1} V_2^{k_2}$ for all $\boldsymbol{k} = (k_1, k_2) \in \mathbb{N}^2$.
For $m+1 \leq n$: Now let for $m < n$, we have $\mathcal{H} = \oplus_{A \subseteq I_m} \mathcal{H}_A$, where for each non-empty subset A of I_m

$$\mathcal{H}_A = \sum_{k_i \in \mathbb{N}} \oplus (\prod_{i \in A} V_i^{k_i})(\bigcap_{k_j \in \mathbb{N}} (\prod_{j \in I_m \setminus A} V_j^{k_j}) \mathcal{W}_\alpha),$$

and for $A = \phi \subseteq I_m$,

$$\mathcal{H}_A = \bigcap_{k_1, \ldots, k_m \in \mathbb{N}} V_1^{k_1} \cdots V_m^{k_m} \mathcal{H}.$$

We claim that

$$\mathcal{H} = \sum_{A \subseteq I_{m+1}} \oplus \mathcal{H}_A.$$

Since $\mathcal{W}_A$ is V_{m+1}-reducing subspace for all none-empty $A \subseteq I_m$, we have

$$\begin{aligned}\mathcal{W}_A &= \sum_{k_{m+1} \in \mathbb{N}} \oplus V_{m+1}^{k_{m+1}} (\mathcal{W}_A \ominus V_{m+1} \mathcal{W}_A) \oplus \bigcap_{k_{m+1} \in \mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A \\ &= \sum_{k_{m+1} \in \mathbb{N}} \oplus V_{m+1}^{k_{m+1}} (\bigcap_{i \in A} \mathcal{W}_i \bigcap \mathcal{W}_j) \oplus \bigcap_{k_{m+1} \in \mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A,\end{aligned}$$

and hence it follows that

$$\begin{aligned}\mathcal{H}_A &= \sum_{k_i \in \mathbb{N}} \oplus (\prod_{i \in A} V_i^{k_i}) \Big(\sum_{k_{m+1} \in \mathbb{N}} \oplus V_{m+1}^{k_{m+1}} (\bigcap_{i \in A} \mathcal{W}_i \bigcap \mathcal{W}_j) \oplus \bigcap_{k_{m+1} \in \mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A \Big) \\ &= \sum_{k_i, k_{m+1} \in \mathbb{N}} \oplus (\prod_{i \in A} V_i^{k_i} V_{m+1}^{k_{m+1}}) (\bigcap_{i \in A} \mathcal{W}_i \bigcap \mathcal{W}_j) \oplus \sum_{k_i \in \mathbb{N}} \oplus (\prod_{i \in A} V_i^{k_i}) (\bigcap_{k_{m+1} \in \mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A).\end{aligned}$$

Applying again the Wold decomposition to the isometry V_{m+1} on $\mathcal{H}$, we have

$$\mathcal{H} = \sum_{k_{m+1} \in \mathbb{N}} \oplus V_{m+1}^{k_{m+1}} \mathcal{W}_{m+1} \oplus \bigcap_{k_{m+1} \in \mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{H},$$

and hence for $A = \phi \subseteq I_m$,

$$\begin{aligned}\mathcal{H}_A &= \bigcap_{k_1, \ldots, k_m \in \mathbb{N}} V_1^{k_1} \cdots V_m^{k_m} \mathcal{H} \\ &= \bigcap_{k_1, \ldots, k_m \in \mathbb{N}} V_1^{k_1} \cdots V_m^{k_m} (\sum_{k_{m+1} \in \mathbb{N}} \oplus V_{m+1}^{k_{m+1}} \mathcal{W}_{m+1} \oplus \bigcap_{k_{m+1} \in \mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{H}) \\ &= \sum_{k_{m+1} \in \mathbb{N}} \oplus V_{m+1}^{k_{m+1}} (\bigcap_{k_1, \ldots, k_m \in \mathbb{N}} V_1^{k_1} \cdots V_m^{k_m} \mathcal{W}_{m+1}) \oplus \bigcap_{k_1, \ldots, k_m, k_{m+1} \in \mathbb{N}} V_1^{k_1} \cdots V_m^{k_m} V_{m+1}^{k_{m+1}} \mathcal{H}.\end{aligned}$$

Consequently,

$$\mathcal{H} = \sum_{A \subseteq I_{m+1}} \oplus \mathcal{H}_A.$$

It follows immediately from the above orthogonal decomposition of $\mathcal{H}$ that $V_i|_{\mathcal{H}_A}$ is a shift for all $i \in A$ and unitary for all $i \in I_n \setminus A$.
The uniqueness part follows from the uniqueness of the classical Wold decomposition of isometries and the canonical construction of the present orthogonal decomposition. This completes the proof. ∎

Note that if $n = 2$, then (3.1) yields a new proof of Slocinski's Wold-type decomposition of a pair of doubly commuting isometries.

The following corollary is an n-variables analogue of the wandering subspace representations of pure isometries (that is, shift operators) on Hilbert spaces.

Corollary 3.2. *Let $V = (V_1, \dots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting shift operators on $\mathcal{H}$. Then*

$$\mathcal{W} = \bigcap_{i=1}^{n} ran(I - V_i V_i^*),$$

is a wandering subspace for V and

$$\mathcal{H} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus V^{\boldsymbol{k}} \mathcal{W}.$$

Proof. Let us note first that the given condition is equivalent to

$$\bigcap_{k \in \mathbb{N}} V_i^k \mathcal{H} = \{0\},$$

for all $1 \leq i \leq n$. Then the result readily follows from the proof of Theorem 3.1. ∎

Recall that a pair of n-tuples $V = (V_1, \dots, V_n)$ and $W = (W_1, \dots, W_n)$ on $\mathcal{H}$ and $\mathcal{K}$, respectively, are said to be unitarily equivalent if there exists a unitary map $U : \mathcal{H} \to \mathcal{K}$ such that $UV_i = W_i U$ for all $1 \leq i \leq n$.

The following corollary is a generalization of Theorem 1 in [9].

Theorem 3.3. *Let $V = (V_1, \dots, V_n)$ be an n-tuple ($n \geq 2$) of commuting isometries on $\mathcal{H}$. Then the following conditions are equivalent:*

(i) There exists a wandering subspace $\mathcal{W}$ for V such that

$$\mathcal{H} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus V^{\boldsymbol{k}} \mathcal{W}.$$

(ii) V_m is a shift for all $m = 1, \dots, n$.

(iii) There exists $m \in \{1, \dots, n\}$ such that V_m is a shift and the wandering subspace for V_m is given by

$$\mathcal{W}_m = \sum_{\boldsymbol{k} \in \mathbb{N}^n, k_m = 0} \oplus \quad V^{\boldsymbol{k}} (\bigcap_{i=0}^{n} \mathcal{W}_i).$$

(iv) $\mathcal{W} := \bigcap_{i=1}^n \mathcal{W}_i$ is a wandering subspace for V and that $\mathcal{H} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus V^{\boldsymbol{k}} \mathcal{W}$.

(v) V is unitarily equivalent to $M_z = (M_{z_1}, \dots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ for some Hilbert space $\mathcal{E}$ with $dim \mathcal{E} = dim \mathcal{W}$.

Proof. (i) implies (ii) : That V_m is a shift, for all $1 \leq m \leq n$, follows from the fact that

$$\mathcal{H} = \sum_{k \in \mathbb{N}} \oplus V_m^k \Big(\sum_{\boldsymbol{k} \in \mathbb{N}^n, \, k_m = 0} \oplus \quad V^{\boldsymbol{k}} \mathcal{W} \Big).$$

Now let $h \in \mathcal{H}$ and that

$$h = \sum_{i=0}^{\infty} V_m^i f_i. \qquad (f_i \in \sum_{\boldsymbol{k} \in \mathbb{N}^n, \, k_m = 0} \oplus \quad V^{\boldsymbol{k}} \mathcal{W})$$

It follows that for all $l \neq m$,

$$V_l V_m^* h = V_l (\sum_{i=1}^{\infty} V_m^{i-1} f_i) = \sum_{i=1}^{\infty} V_m^{i-1} (V_l f_i) = V_m^* (\sum_{i=0}^{\infty} V_m^i (V_l f_i)) = V_m^* V_l (\sum_{i=0}^{\infty} V_m^i f_i),$$

that is, V is doubly commuting.

(ii) implies (iii): By Corollary 3.1 we have

$$\mathcal{H} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus V^{\boldsymbol{k}} (\bigcap_{i=1}^{n} \mathcal{W}_i) = \sum_{k \in \mathbb{N}} \oplus V_m^k \Big(\sum_{\boldsymbol{k} \in \mathbb{N}^n, \, k_m = 0} \oplus \quad V^{\boldsymbol{k}} (\bigcap_{i=1}^{n} \mathcal{W}_i) \Big),$$

and hence (iii) follows.

(iii) implies (iv): Since V_m is a shift with the wandering subspace

$$\mathcal{W}_m = \sum_{\boldsymbol{k} \in \mathbb{N}^n, \, k_m = 0} \oplus \quad V^{\boldsymbol{k}} (\bigcap_{i=0}^{n} \mathcal{W}_i),$$

we infer that

$$\mathcal{H} = \sum_{k \in \mathbb{N}} \oplus V_m^k \mathcal{W}_m = \sum_{k \in \mathbb{N}} \oplus V_m^k \Big(\sum_{\boldsymbol{k} \in \mathbb{N}^n, \, k_m = 0} \oplus \quad V^{\boldsymbol{k}} (\bigcap_{i=0}^{n} \mathcal{W}_i) \Big) = \sum_{k \in \mathbb{N}} \oplus V_m^k (\bigcap_{i=1}^{n} \mathcal{W}_i),$$

and hence (iv) follows.

(iv) implies (v): Let $\mathcal{E} = \bigcap_{i=1}^n \mathcal{W}_i$. Define the unitary operator

$$U : \mathcal{H} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus V^{\boldsymbol{k}} (\bigcap_{i=1}^{n} \mathcal{W}_i) \longrightarrow H^2_{\mathcal{E}}(\mathbb{D}^n) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus z^{\boldsymbol{k}} \mathcal{E},$$

by $U(V^{\boldsymbol{k}} \eta) = z^{\boldsymbol{k}} \eta$ for all $\eta \in \mathcal{E}$ and $\boldsymbol{k} \in \mathbb{N}^n$. It is obvious that $U V_i = M_{z_i} U$ for all $i = 1, \ldots, n$.

That (v) implies (i) is trivial.

This concludes the proof of the theorem. ∎

As we indicated earlier, the Wold decomposition theorem may be regarded as a very powerful tool from which a number of classical results may be deduced. In [7] we will use techniques developed in this paper to obtain a complete classification of doubly commuting invariant subspaces of the Hardy space over the polydisc.

Finally, we point out that all results of this paper are also valid for a family of doubly commuting isometries.

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