Fourier Series: Infinite Sequences

$$\{a_t : t = \ldots, -1, 0, 1, \ldots\}$$

is an infinite sequence

- a_t's are real or complex-valued variables
- assume $\sum_t |a_t|^2 < \infty$

Discrete Fourier Transform

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft}, \qquad -\infty < f < \infty$$

- f called frequency.
- $A(\cdot)$ called Fourier series of $\{a_t\}$

DFT : Periodicity

For any integer j,

$$
\begin{aligned}
A(f+j) &= \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi(f+j)t} \\
\text{(why ?)} &= \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi f t} \\
&= A(f)
\end{aligned}
$$

DFT : Infinite Sequences

- since $A(\cdot)$ is periodic with unit period, can take

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi f t}, \qquad |f| \leq 1/2$$

and also

$$\int_{-1/2}^{1/2} A(f) e^{i2\pi f t}\, df = a_t, \qquad t = \ldots, -1, 0, 1, \ldots$$

- Notation: $\{a_t\} \longleftrightarrow A(\cdot)$

Convolution : Infinite Sequences

- given $\{a_t\} \longleftrightarrow A(\cdot)$ & $\{b_t\} \longleftrightarrow B(\cdot)$, define

$$a{*}b_t \equiv \sum_{u=-\infty}^{\infty} a_u b_{t-u}, \qquad t = \ldots, -1, 0, 1, \ldots$$

and sequence $\{a * b_t\}$ is convolution of $\{a_t\}$ & $\{b_t\}$.

Note: '$a * b$' is just a variable (like 'a' or 'b')

Convolution : Infinite Sequences

$$\sum_{t=-\infty}^{\infty} a * b_t e^{-i2\pi f t} = A(f)B(f);$$

i.e., $\{a * b_t\} \longleftrightarrow A(\cdot)B(\cdot)$

$$a^* \star b_t \equiv \sum_{u=-\infty}^{\infty} a_u^* b_{u+t} \qquad t = \ldots, -1, 0, 1, \ldots,$$

for which $\{a^* \star b_t\} \longleftrightarrow A^*(\cdot)B(\cdot)$.

Discrete Fourier Transform : Filtering

Concept of Filter:

- $\{b_t\}$ is input to filter, $\{a_t\}$ is (impulse response sequence for) filter
- $\{a * b_t\}$ is output from filter

Flow diagram for filtering:

$$\{b_t\} \longrightarrow \boxed{\{a_t\}} \longrightarrow \{a * b_t\}$$

$$\{b_t\} \longrightarrow \boxed{A(\cdot)} \longrightarrow \{a * b_t\}$$

DFT : Finite Sequences

- $\{a_t : t = 0, 1, \ldots, N - 1\}$ is a finite sequence of real or complex valued variables

- DFT of $\{a_t\}$:

$$A_k \equiv \sum_{t=0}^{N-1} a_t e^{-i2\pi tk/N}, \qquad k = 0, 1, \ldots, N-1$$

A_k associated with $f_k \equiv k/N$

DFT : Finite Sequences

$\{A_k : k = \ldots, -1, 0, 1, \ldots\}$ periodic

$$A_{k+nN} = \sum_{t=0}^{N-1} a_t e^{-i2\pi t(k+nN)/N} = \sum_{t=0}^{N-1} a_t e^{-i2\pi tk/N} = A_k$$

$$\frac{1}{N}\sum_{k=0}^{N-1} A_k e^{i2\pi tk/N} = a_t, \qquad t = 0, 1, \ldots, N-1;$$

left-hand side called inverse DFT of $\{A_k\}$

Convolution of Finite Sequences

Given $\{a_t\} \longleftrightarrow \{A_k\}$ & $\{b_t\} \longleftrightarrow \{B_k\}$, define

$$a*b_t \equiv \sum_{u=0}^{N-1} a_u b_{t-u \bmod N}, \qquad t = 0, 1, \ldots, N-1;$$

$$\{a * b_t\} \longleftrightarrow \{A_k B_k\}$$

$$\{b_t\} \longrightarrow \boxed{\{A_k\}} \longrightarrow \{a * b_t\}$$

Periodised Filters

Let $\{b_t : t = 0, \ldots, N-1\}$ and filter be $\{a_t : t = \ldots, -1, 0, 1, \ldots\}$.

$$c_t \equiv \sum_{v=-\infty}^{\infty} a_v b_{t-v \bmod N}, \qquad t = 0, \ldots, N-1$$

Rewrite:

$$c_t \equiv \sum_{u=0}^{N-1} a_u^{\circ} b_{t-u \bmod N} \qquad t = 0, \ldots, N-1$$

where $a_u^{\circ} = \sum_{n=-\infty}^{\infty} a_{u+nN}$.

Discrete Fourier Transform : Periodised Filters

In summary: periodized filter $\{a_t^\circ\}$ formed by chopping $\{a_t\}$ into finite sequences of length N:

$$a_u^\circ = \sum_{n=-\infty}^{\infty} a_{u+nN} \quad u = 0, \ldots, N-1$$

Then $c_t \equiv a * b_t \equiv a^\circ * b_t$.

Periodised Filters: Example

Length N periodization of infinite sequence

$$a_t = \begin{cases} 1/2, & \text{t=0;} \\ 1/4, & \text{t=+1, -1;} \\ 0, & otherwise. \end{cases} \quad \text{gives} \quad a_t^\circ = \begin{cases} 1/2, & \text{t=0;} \\ 1/4, & \text{t=1\&N-} \\ 0, & \text{t=2, \ldots ,} \end{cases}$$

Q: if $\{a_t\} \longleftrightarrow A(\cdot)$, what is DFT of $\{a_t^\circ\}$?

Periodised Filters: Example

A: $\{a_t^\circ\} \longleftrightarrow \{A(\frac{k}{N}) : k = 0, \ldots, N-1\}$;
i.e. periodization in time domain equivalent to sampling in frequency domain.

So:

$$c_t = \sum_{v=-\infty}^{\infty} a_v b_{t-v \bmod N}, \qquad t = 0, \ldots, N-1$$

as

$$\{b_t\} \longrightarrow \boxed{A(\tfrac{k}{N})} \longrightarrow \{c_t\}$$

Analysis/Synthesis of Time Series

- can analyze $\mathbf{X}$ with respect to transform $\mathcal{O}$:

$$\mathbf{O} \equiv \mathcal{O}\mathbf{X} = \begin{bmatrix} \vdots \\ \mathcal{O}_{j\bullet}^T \\ \vdots \end{bmatrix} \mathbf{X} = \begin{bmatrix} \vdots \\ \mathcal{O}_{j\bullet}^T \mathbf{X} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \langle \mathbf{X}, \mathcal{O}_{j\bullet} \rangle \\ \vdots \end{bmatrix}$$

  $\mathbf{O}$ called transform coefficients; jth is $O_j = \langle \mathbf{X}, \mathcal{O}_{j\bullet} \rangle$

- premultiply by $\mathcal{O}^T$ to get $\mathcal{O}^T\mathbf{O} = \mathcal{O}^T\mathcal{O}\mathbf{X} =$

$\mathbf{X}$; i.e.,
can synthesize $\mathbf{X}$ from its transform coefficients $\mathbf{O}$:

$$\mathbf{X} = \mathcal{O}^T\mathbf{O} = \begin{bmatrix} \cdots & \mathcal{O}_{j\bullet} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ O_j \\ \vdots \end{bmatrix} = \sum_{j=0}^{N-1} O_j \mathcal{O}_{j\bullet}$$

key to additive decomposition: $O_j \mathcal{O}_{j\bullet}$ is $N \times 1$ vector

- energy preservation (isometry):

$$\begin{aligned} \mathcal{E}_{\mathbf{O}} \equiv \|\mathbf{O}\|^2 = \mathbf{O}^T\mathbf{O} &= (\mathcal{O}\mathbf{X})^T\mathcal{O}\mathbf{X} \\ &= \mathbf{X}^T\mathcal{O}^T\mathcal{O}\mathbf{X} = \mathbf{X}^T\mathbf{X} = \|\mathbf{X}\|^2 \end{aligned}$$

key to analysis of variance

- can also show (Exercise):

$$\|\mathbf{X}\|^2 = \sum_{j=0}^{N-1} \|O_j \mathcal{O}_{j\bullet}\|^2$$

$O_j \mathcal{O}_{j\bullet}$ is jth series in additive decomposition

Projection Theorem: I

- consider an approximation to $\mathbf{X}$ of form

$$\widehat{\mathbf{X}} = \sum_{j=0}^{N'-1} \alpha_j \mathcal{O}_{j\bullet}, \quad N' < N$$

- want to pick α_j's so that approximation error

$$\mathbf{e} \equiv \mathbf{X} - \widehat{\mathbf{X}}$$