The Haar DWT

- to see scale/location aspect, consider Haar $\mathcal{W}$
- formulation of first $\frac{N}{2}$ rows:
  - row $j = 0$: $\big[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \underbrace{0,\ldots,0}_{N-2 \text{ zeros}}\big] \equiv \mathcal{W}_{0\bullet}^T$

    note: $\|\mathcal{W}_{0\bullet}\|^2 = \frac{1}{2} + \frac{1}{2} = 1$, as required
  - row $j = 1$: $\big[0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \underbrace{0,\ldots,0}_{N-4 \text{ zeros}}\big] \equiv \mathcal{W}_{1\bullet}^T$

    note: $\mathcal{W}_{0\bullet}$ & $\mathcal{W}_{1\bullet}$ are orthonormal pair
  - row $j = \frac{N}{2} - 1$: $\big[\underbrace{0,\ldots,0}_{N-2 \text{ zeros}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\big] \equiv \mathcal{W}_{\frac{N}{2}-1\bullet}^T$
  - can express transpose of jth row as

$$\mathcal{W}_{j\bullet} = \mathcal{T}^{2j}\mathcal{W}_{0\bullet}, \;\; j = 0, \ldots \tfrac{N}{2} - 1$$

- first $\frac{N}{2}$ rows form orthonormal set of $\frac{N}{2}$ vectors
- yields $\frac{N}{2}$ wavelet coefficients of 'scale 1,' location $2j$

- formulation of next $\frac{N}{4}$ rows:
  - $j = \frac{N}{2}$: $\big[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \underbrace{0,\ldots,0}_{N-4 \text{ zeros}}\big] \equiv \mathcal{W}_{\frac{N}{2}\bullet}^T$

    note: $\|\mathcal{W}_{\frac{N}{2}\bullet}\|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$, as required

    note: $\mathcal{W}_{\frac{N}{2}\bullet}$ & $\mathcal{W}_{0\bullet}$ etc. are orthonormal pair
  - $j = \frac{N}{2} + 1$: $\big[0, 0, 0, 0, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \underbrace{0,\ldots,0}_{N-8 \text{ zeros}}\big] \equiv \mathcal{W}_{\frac{N}{2}+1\bullet}^T$

    note: $\mathcal{W}_{\frac{N}{2}\bullet}$ & $\mathcal{W}_{\frac{N}{2}+1\bullet}$ are orthonormal pair
  - row $j = \frac{3N}{4} - 1$: $\big[\underbrace{0,\ldots,0}_{N-4 \text{ zeros}}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big] \equiv \mathcal{W}_{\frac{3N}{4}-1\bullet}^T$
  - can express transpose of row $j = \frac{N}{2} + k$ as

$$\mathcal{W}_{\frac{N}{2}+k\bullet} = \mathcal{T}^{4k}\mathcal{W}_{\frac{N}{2}\bullet}, \;\; k = 0, \ldots \tfrac{N}{4} - 1$$

- first $\frac{3N}{4}$ rows form orthonormal set of $\frac{3N}{4}$ vectors
- $\frac{N}{4}$ rows from $j = \frac{N}{2}$ to $\frac{3N}{4} - 1$ yield
  $\frac{N}{4}$ wavelet coefficients of 'scale 2,' location $4j$

- formulation of next $\frac{N}{8}$ rows:
  - $j = \frac{3N}{4}$: $\big[\underbrace{-\frac{1}{\sqrt{8}}, \ldots, -\frac{1}{\sqrt{8}}}_{4 \text{ of these}}, \underbrace{\frac{1}{\sqrt{8}}, \ldots, \frac{1}{\sqrt{8}}}_{4 \text{ of these}}, \underbrace{0,\ldots,0}_{N-8 \text{ zeros}}\big] \equiv \mathcal{W}_{\frac{3N}{4}\bullet}^T$

    note: $\|\mathcal{W}_{\frac{3N}{4}\bullet}\|^2 = 8 \cdot \frac{1}{8} = 1$, as required

- can express transpose of row $j = \frac{3N}{4} + k$ as

$$\mathcal{W}_{\frac{3N}{4}+k\bullet} = \mathcal{T}^{8k}\mathcal{W}_{\frac{3N}{4}\bullet}, \;\; k = 0, \ldots \tfrac{N}{8} - 1$$

- $\frac{N}{8}$ rows starting with $j = \frac{3N}{4}$ yield
  $\frac{N}{8}$ wavelet coefficients of 'scale 4,' location $8k$
- …and so it goes until finally we come to:
  - $j = N - 2$: $\big[\underbrace{-\frac{1}{\sqrt{N}}, \ldots, -\frac{1}{\sqrt{N}}}_{\frac{N}{2}\text{ of these}}, \underbrace{\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}}_{\frac{N}{2}\text{ of these}}\big] \equiv \mathcal{W}^T_{N-2\bullet}$
  - associated with wavelet coefficient of scale $\frac{N}{2}$
  - $j = N - 1$: $\big[\underbrace{\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}}_{N\text{ of these}}\big] \equiv \mathcal{W}^T_{N-1\bullet}$

    note: $\|\mathcal{W}_{N-2\bullet}\|^2 = \|\mathcal{W}_{N-1\bullet}\|^2 = N \cdot \frac{1}{N} = 1$
  - associated with coefficient of scale N
- set of N orthonormal vectors in all

Problem 1: Verify that $\mathcal{W}$ is an orthonormal matrix. Can you construct another $\mathcal{W}$ from this ?