Basics and Notation:

- $i \equiv \sqrt{-1}$, so $z = x + iy$ is complex-valued
  (note: '$\equiv$' means 'equal by definition')
- $z^* \equiv x - iy$
- $|z| \equiv \sqrt{x^2 + y^2}$ so $|z|^2 = x^2 + y^2 = |z^*|^2 = zz^*$
- $e^{ix} \equiv \cos(x) + i\sin(x)$ (Euler's relationship)
  - $|e^{ix}|^2 = 1$ because $\cos^2(x) + \sin^2(x) = 1$
  - $e^{i(x+y)} = e^{ix}e^{iy}$ – just expand out both sides
  - $(e^{ix})^n = e^{inx}$ for integer n (de Moivre's theorem)
  - $\int e^{ix}\,dx = \frac{e^{ix}}{i}$ because
    $\int \cos(x) + i\sin(x)\,dx = \sin(x) - i\cos(x) = \frac{i\sin(x)+\cos(x)}{i}$
  - since $e^{-ix} = \cos(x) - i\sin(x)$, have
    $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$
  - $e^{\pm i\pi} = -1$ (trivial, but useful!)
- can write $z = |z|e^{i\theta}$ (polar representation)
  - $|z|$ is magnitude of z ($|z|$ is nonnegative)
  - $\theta = \arg(z)$ is argument of z (defined if $|z| > 0$);
    $\theta =$ angle between positive x axis & line to (x, y)
  - make convention that $-\pi < \theta \leq \pi$;
    $\theta > 0$ when (x, y) plotted above x axis
- place real-valued time series in column vector:

  $$\mathbf{X} = \begin{bmatrix} X_0, & X_1, & \ldots, & X_{N-1} \end{bmatrix}^T$$

  note: 'T' denote transpose
- inner product: $\langle \mathbf{X}, \mathbf{Y} \rangle \equiv \mathbf{X}^T\mathbf{Y} = \sum_{t=0}^{N-1} X_t Y_t$
- squared norm: $\|\mathbf{X}\|^2 \equiv \langle \mathbf{X}, \mathbf{X} \rangle = \sum_{t=0}^{N-1} X_t^2 \equiv \mathcal{E}_{\mathbf{X}}$;
- For any matrix $\mathcal{O}$
  - $\mathcal{O}_{j\bullet}^T$ denotes jth row vector
  - $\mathcal{O}_{\bullet k}$ denotes kth column vector

Discrete Fourier Transform for infinite sequences: $\{a_t : t = \ldots, -1, 0, 1, \ldots\}$, a_t's are real or complex-valued variables and assume $\sum_t |a_t|^2 < \infty$. The DFT of a_t is

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft}, \qquad -\infty < f < \infty$$

1. Can you tell when will $|A(\cdot)|$ be large ?
2. If a_t is real then show that $A(-f) = A*(f)$.
3. Show that for $t \in \mathbb{Z}$, $a_t = \int_{-1/2}^{1/2} A(f) e^{i2\pi ft}\, df$
4. Suppose b_t is another time series,
   (a) show that $\sum_{t=-\infty}^{\infty} a * b_t e^{-i2\pi ft} = A(f)B(f)$
   (b) A related concept is complex cross-correlation:$a^* \star b_t \equiv \sum_{u=-\infty}^{\infty} a_u^* b_{u+t} \qquad t = \ldots, -1, 0, 1, \ldots.$ Show that $\{a^* \star b_t\} \longleftrightarrow A^*(\cdot)B(\cdot)$. What do we get when $a = b$?

Discrete Fourier Transform for finite sequences: $\{a_t : t = \ldots, -1, 0, 1, \ldots, N-1\}$, a_t's are real or complex-valued variables and assume $\sum_t |a_t|^2 < \infty$. The DFT of a_t is

$$A_k \equiv \sum_{t=0}^{N-1} a_t e^{-i2\pi \frac{k}{N} t}, \qquad k = 0, 1, \ldots, N-1$$

1. Can you relate $A(\cdot)$ and $A.$?
2. Show that for $t = 0, 1, \ldots, N-1$, $a_t = \frac{1}{N} \sum_{k=0}^{N-1} A_k e^{i2\pi t \frac{k}{N}}$
3. Suppose b_t is another finite time series,
   (a) if $\{a_t\} \longleftrightarrow \{A_k\}$ & $\{b_t\} \longleftrightarrow \{B_k\}$, then show that

$$\sum_{t=0}^{N-1} a_t b_t^* = \frac{1}{N} \sum_{k=0}^{N-1} A_k B_k^*$$

   What do you get when $a = b$? Can you give a generalisation of this problem for the infinite time series case ?

   (b) Show that $\{a * b_t\} \longleftrightarrow \{A_k B_k\}$.
   (c) Suitably define the concept of complex cross-correlation, $a^* \star b_t$ in the finite case and find its DFT.

Periodised (Circular) Filter: Let $\{b_t : t = 0, \ldots, N-1\}$ be a finite sequence, and use $\{a_t : t = \ldots, -1, 0, 1, \ldots\}$ to form

$$c_t \equiv \sum_{v=-\infty}^{\infty} a_v b_{t-v \bmod N}, \qquad t = 0, \ldots, N-1$$

(a) Show that $c_t = \sum_{u=0}^{N-1} a_u^\circ b_{t-u \bmod N}$ with $a_u^\circ = \sum_{n=-\infty}^{\infty} a_{u+nN}$.
(b) Suppose

$$a_t = \begin{cases} 1/2, & \text{t=0;} \\ 1/4, & \text{t=+ 1,-1;} \\ 0, & \textit{otherwise.} \end{cases}$$

find a_t° and its DFT.

(c) In general show that if $\{a_t\} \longleftrightarrow A(\cdot)$ then$\{a_t^\circ\} \longleftrightarrow \{A(\frac{k}{N}) : k = 0, \dots, N-1\}$; Justify the statment: "periodisation in time domain is equivalent to sampling in frequency domain."

(d) Suppose $\sum_{t=-\infty}^{\infty} |a_t|^2 = 1$. does it imply $\sum_{t=0}^{N-1} |a_t^\circ|^2 = 1$?

Orthonormal Transformations: A real matrix $\mathcal{O}$ is called Orthonormal if $\mathcal{O}^T\mathcal{O} = I$

1. Let $\mathbf{X}$ be as in notation and $A = \mathcal{O}X$. Show that $\|\mathbf{X}\|^2 = \sum_{j=0}^{N-1} \|a_j \mathcal{O}_{j\bullet}\|^2$

2. Suppose $\widehat{\mathbf{X}} = \sum_{j=0}^{N'-1} \alpha_j \mathcal{O}_{j\bullet}$, $N' < N$ for some $\alpha_j \in \mathbb{R}$.and $\mathbf{e} \equiv \mathbf{X} - \widehat{\mathbf{X}}$. Show that $\|\mathbf{e}\|^2 = \sum_{j=0}^{N'-1} (a_j - \alpha_j)^2 + \sum_{j=N'}^{N-1} a_j^2$

Unitary Transformations:A real(complex) matrix $\mathcal{U}$ is called Unitary if $\mathcal{U}^H\mathcal{U} = I$.

1. Let $\mathcal{F}$ be $N \times N$ matrix with elements $e^{-i2\pi tk/N}/\sqrt{N}$, $0 \le k, t \le N-1$, show that $\mathcal{F}$ is unitary.

2. Let N be even. Suppose $\mathbf{X}$ be as in notation, $F = \mathcal{F}\mathbf{X}$

   (a) Show that $F_{N-k}^* = F_k$, $0 < k < N/2$.

   (b) Show that $X = \overline{X}\mathbf{1} + \sum_{1 \le k \le \frac{N}{2}} \mathcal{D}_{\mathcal{F},k}$ where

$$\mathcal{D}_{\mathcal{F},k} = \begin{cases} F_k \mathcal{F}_{k\bullet} + F_{N-k}\mathcal{F}_{N-k\bullet} & 1 < k < \frac{N}{2} \\ F_{\frac{N}{2}} \mathcal{F}_{\frac{N}{2}} & k = \frac{N}{2} \end{cases}$$

   (c) Suppose $F_k = A_k - iB_k$, then show that the t term in $\mathcal{D}_{\mathcal{F},k}$ is given by

$$\mathcal{D}_{\mathcal{F},k,t} = \frac{2}{\sqrt{N}} \left[A_k \cos(2\pi f_k t) + B_k \sin(2\pi f_k t)\right]$$

Invariance under Circular Shifts Let $\mathcal{T}$ & $\mathcal{T}^{-1}$ be $N \times N$ matrices such that

$$\mathbf{X} = \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-2} \\ X_{N-1} \end{bmatrix} \Longrightarrow \mathcal{T}\mathbf{X} = \begin{bmatrix} X_{N-1} \\ X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-2} \end{bmatrix} \quad \& \quad \mathcal{T}^{-1}\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-2} \\ X_{N-1} \\ X_0 \end{bmatrix}$$

1. Suppose $\mathcal{T}$ & $\mathcal{T}^{-1}$ contain just 0's and 1's & $\mathcal{T}\mathcal{T}^{-1} = I_N$. Define $\mathcal{T}^2\mathbf{X} = \mathcal{T}\mathcal{T}\mathbf{X}$, $\mathcal{T}^{-2}\mathbf{X} = \mathcal{T}^{-1}\mathcal{T}^{-1}\mathbf{X}$, etc. Show that the kth Fourier coefficient of $\mathcal{T}^m\mathbf{X}$ is $F_k \exp(-i2\pi mk/N)$

2. Conclude that $\mathbf{X}$ & $\mathcal{T}^m\mathbf{X}$ have same power spectrum

3. Show that if $\mathbf{X}$ has detail $\mathcal{D}_{\mathcal{F},k}$, then $\mathcal{T}^m\mathbf{X}$ has detail $\mathcal{T}^m\mathcal{D}_{\mathcal{F},k}$

4. Discuss why isn't ODFT used instead of DFT?