Orthonormal Wavelet: A function $\psi : \mathbb{R} \to \mathbb{R}$ such that

1. $\psi(t)$ tends to zero faster than any power of t as $t \to \infty$
2. ψ possesses continuous derivatives upto order N, for some positive integer N.
3. For all integers j and n, let
$$\psi_{jn}(t) = 2^{\frac{j}{2}}\psi(2^j t - n).$$
Then "suitable" functions can be expanded uniquely in a series of ψ_{jn}:
$$f(t) = \sum_{j,n=-\infty}^{\infty} c_{jn}\psi_{jn}(t).$$
4. The coefficients c_{jn} above are given by
$$c_{jn} = \int_{-\infty}^{\infty} f(t)\overline{\psi_{jn}(t)}dt.$$

Notes

1. Please try to solve the problems in the next page.
2. If you had some confusion with today's lecture then please send me an email. I will try to clarify next time.
3. If you give me hand written/typed or electronic solutions then I shall try to give you feedback on them.

Problem Set 1

1. Consider $x(\cdot)$ a 'signal', i.e. a real-valued function of t defined over real axis and its average value of $x(\cdot)$ over $[a, b]$:
$$\frac{1}{b-a}\int_a^b x(u)\,du \equiv \alpha(a,b)$$
   (a) Suppose we divide $[a, b]$ into N equal parts and x is piece-wise constant on each part, what does $\alpha(a, b)$ correspond to ?
   (b) Let
$$A(\lambda,t) = \alpha(t-\tfrac{\lambda}{2}, t+\tfrac{\lambda}{2}) \text{ and } D(\lambda,t) = A(\lambda, t+\frac{\lambda}{2}) - A(\lambda, t-\frac{\lambda}{2}).$$
   For given choices of λ: what will a plot of $A(\lambda, t)$ and $D(\lambda, t)$ as a function of t tells us about $x(\cdot)$?
   (c) Can you find a function $\psi : \mathbb{R} \to \mathbb{R}$ such that: $\int_{-\infty}^{\infty} x(u)\psi_{\lambda,t}(u)$ is proportional to $D(\lambda, t)$ with $\psi_{\lambda,t} : \mathbb{R} \to \mathbb{R}$ being obtained through with a suitable dilation and translation of ψ
   (d) Generate/download/obtain a signal $x(\cdot)$ and for fixed choices of λ generate plots of $A(\lambda, t)$, $D(\lambda, t)$
   (e) If we know $W(\lambda, t) = \int_{-\infty}^{\infty} x(u)\psi_{\lambda,t}(u)$ then do you know a way of recovering $x(\cdot)$?

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is periodic with period P, can you write down the Fourier series expansion (similar to that of period 1) for the same ? What happens to the formula when $P \to \infty$?

3. Consider $f, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ given by
$$f(t) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi n t}, g_1(t) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n t)}{n}, \text{ and, } g_2(t) = \sum_{n=1}^{\infty} \frac{(-1)^n \sin(2\pi n t)}{n}$$
.
   (a) Show that $g_1 = f$ with $b_n = \frac{1}{2n}$ and find b_n which will make $g_2 = f$.
   (b) Show that f is of period 1 and interpret b_n interms of f.

   with a suitable dilation and translation obtain

4. Suppose f is as in the definition of the orthonormal wavelet. Solve the following:
   (a) Can you interpret the term orthonormal ?
   (b) Rewrite $f = \sum_{j=-\infty}^{\infty} S_j$, with $S_j = \sum_{n=-\infty}^{\infty} c_{jn}\psi_{jn}$. Interpret S_j in terms of f for positive, zero, and negative j.
   (c) Suppose f is given as above and has continuous derivatives of order k at a point a then can you say something about c_{jn} for $j, n \in \mathbb{Z}$?
   (d) Can you find a way to see that g_1, g_2 given in the above exercise will have a different expansion in the definition of the orthonormal wavelet ?