Parabolic SPDEs and Intermittency

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Contents

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Prelude

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. –attributed to ALBERT EINSTEIN

Introduction. This is a five-lecture introduction to stochastic partial differential equations, or SPDEs for short. Because SPDEs is a huge subject that spans a great amount of both theory and applications, we limit ourselves only to a family of more or less concrete, and interesting, mathematical problems. You can find several pointers to these, and other, aspects of the SPDE literature in the references.

In order to have in mind an example of the sort of object that we plan to study, consider first the following *H*eat *e*quation: We seek to find a function u of a space-time variable $(x\,,t) \in \mathbf{R}_+ \times [0\,,1]$ such that

$$\left[\begin{array}{l} \partial_t u = \frac{1}{2}\partial^2_{xx} u \qquad \text{on } (0\,,\infty)\times[0\,,1];\\ \text{with } u(x\,,0) = \sin(\pi x) \text{ for every } x \in 0\,,1];\\ \text{and } u(0\,,t) = u(1\,,t) \equiv 0 \text{ for all } t > 0. \end{array}\right. \tag{$\mathcal{H}e$}$$

The solution to ($\mathcal{H}e$) is $u(x\,,t) = \sin(\pi x)\exp(-\pi^2 t/2)$, as can be seen by direct differentiation. Figure 1 shows a MATLAB plot of $u(x\,,t)$ for $x \in [0\,,1]$ and $t \in [0\,,1/10]$.

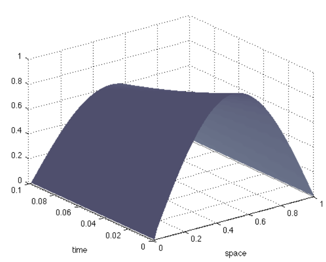

Figure 1: The solutions to ($\mathcal{H}e$); max. height = 1.00

Next let us consider the *S*tochastic *h*eat *e*quation. That is basically the heat equation, but also has a multiplicative random forcing term. Namely,

$$\left[\begin{array}{l} \partial_t u = \frac{1}{2}\partial^2_{xx} u + \lambda u\dot{W} \quad \text{on } (0\,,\infty)\times[0\,,1],\\ \text{with } u(x\,,0) = \sin(\pi x),\\ \text{and } u(0\,,t) = u(1\,,t) \equiv 0, \end{array}\right. \tag{$\mathcal{S}he$}$$

where $\lambda > 0$ is a parameter, and $\dot{W}$ denotes "space-time white noise."[1]

[1] Stated in rough terms, white noise is the continuum equivalent of i.i.d. standard normal random variables. We will discuss it in more detail though, as the proof of its existence require some effort.

Figure 2 shows a simulation of the solution u for all $x \in [0\,,1]$ in the case that $\lambda = 0.1$, where x ranges over all of $[0\,,1]$, and t ranges over the short time interval $[0\,,1/10]$.

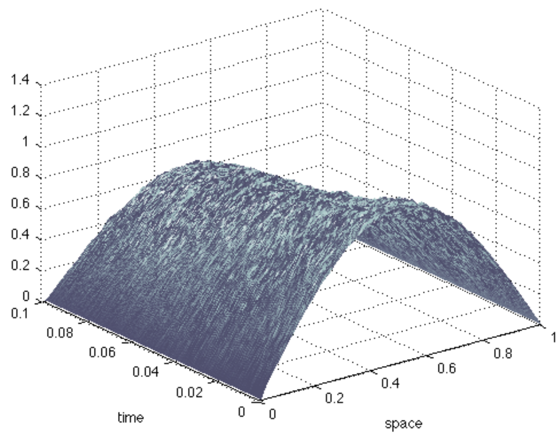

Figure 2: (*She*) with $\lambda = 0.1$; max. height ≈ 1.0197

We can compare Figures 1 and 2 in order to see that the presence of a small amount of noise in (*She*) [here, $\lambda = 0.1$] can have noticable effect on the solution. For instance, the solution to (*She*) is both rougher and taller than the solution to (*He*).

These differences become more pronounced as we add more noise the the system (*She*). For instance, Figure 3 shows a simulation for $\lambda = 2$ where

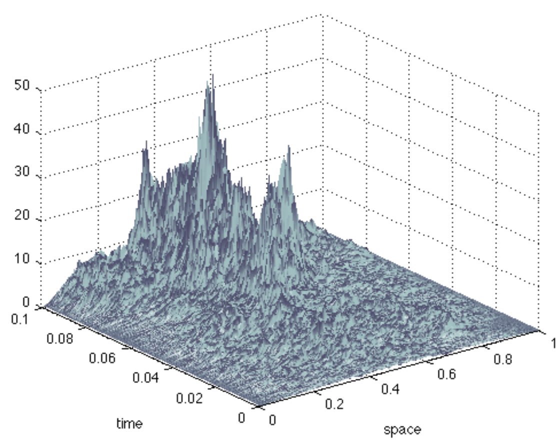

Figure 3: (*She*) with $\lambda = 2$; max. height ≈ 49.1866

$0 \leqslant t \leqslant 1/10$ and $0 \leqslant x \leqslant 1$, once again. In that simulation, the solution is quite rough, and the maximum height of the solution is approximately 50.[2] The tallest peaks are in fact so tall that they dwarf the remainder of the solution. That is why, most of the solution appears to be flat in Figure 3. In fact, that portion is supposed to be a smaller statistical replica of the whole picture—this sort of property is called *multifractal*—see MANDELBROT [35]—though I am not aware of any rigorous ways to state and/or prove the multfractality of u at present.

As you can see from this table, even a modestly-noisy ***She*** is a great deal rougher and taller than its non-random cousin ***He***. This apparent "peakiness" property of ***She*** is usually called *intermittency*, *intermittence*, or *localization*. We will say a few things about this property later on. In the mean time, I mention that two excellent general-audience introductions to intermittency and localization in science are MANDELBROT [35] and ZELDOVITCH ET AL [47].

I conclude with a few sage words by WIGNER [46]:

> *"Let me end on a more cheerful note. The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."*

Notational convention. Suppose $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is a real-valued function of a "time" variable $t \in \mathbf{R}_+$ and a "space" variable $x \in \mathbf{R}$. From now on, I will *always* write $f_t(x)$ in place of $f(t\,,x)$. From an analytic point of view, this convention might seem particularly strange, since we are studying a family of "random PDEs" and f_t usually refers to $\partial_t f$! But as you will soon see this convention makes excellent probabilistic sense. After all, we are interested in the behavior of the stochastic process $\{f_t\}_{t\geqslant 0}$, which typically ends up taking values in some function space.

Numerical simulation. The following material are not covered in the course *per se*, but it would be a shame to not say anything about how one can "produce neat pictures."

Suppose we wish to simulate the solution to the boundary-value problem (***She***). One can indeed prove that, in this case, (***She***) has an a.s.-unique solution u. Moreover, because $u_0(x) \geqslant 0$ for all $x \in [0\,,1]$, one can prove that $u_t(x) \geqslant 0$ for all $t \geqslant 0$ and $x \in [0\,,1]$ a.s. Rather than prove these facts, I refer you to Chapters 1 [§6] and 5 [§5] of the minicourse by DALANG ET AL [17]. We will establish some of these properties for a different family of SPDEs very soon.

[2] One might be tempted to run simulations for very high values of λ. But some careful experimentation will show that the algorithm's errors add up irreparably for values of $\lambda \gg 10$.

Suppose that we wish to simulate the preceding on the time interval $t \in [0\,, T]$, where $T > 0$ is fixed. [In the simulations that we saw earlier, $T = 1/10$.] It stands to reason that if $\Delta t, \Delta x \ll 1$, then the solution ought to satisfy[3]

$$\frac{u_{t+\Delta t}(x) - u_t(x)}{\Delta t} \approx \frac{u_t(x+\Delta x) + u_t(x-\Delta x) - 2u_t(x)}{2(\Delta x)^2} + \frac{\lambda u_t(x)\xi_t(x)}{\Delta t\, \Delta x},$$

where $\xi_t(x)$ is the following "Wiener integral,"

$$\xi_t(x) := \int_{(t,t+\Delta t)\times(x,x+\Delta x)} \dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y.$$

We choose $\Delta t := N^{-2}$ and $\Delta x := N^{-1}$ for an integer $N \gg 1$—as is natural for heat-transfer problems—and obtain

$$u_{t+(T/N^2)}(x) - u_t(x) \approx \frac{u_t(x+(1/N)) + u_t(x-(1/N)) - 2u_t(x)}{2} + \lambda N u_t(x)\xi_t(x),$$

whenever $t \in [0\,, T]$ and $x \in (0\,, 1)$, and $u_{t+N^{-2}}(0) = u_{t+N^{-2}}(1) = 0$. In particular, relabel $t := iN^{-2}$ and $x := jN^{-1}$ to see that

$$u_{(i+1)/N^2}\left(\frac{j}{N}\right) \approx \frac{u_{i/N^2}((j+1)/N) + u_{i/N^2}((j-1)/N)}{2} + \lambda N u_{i/N^2}\left(\frac{j}{N}\right)\xi_{i/N^2}\left(\frac{j}{N}\right).$$

From now on we will ignore the error of approximation, as it is supposed to be small. In other words, we will pretend that "$\approx$" is the same as "$=$." Because the solution is nonnegative for all time, it would then follow that

$$U_i(j) := u_{i/N^2}(j/N), \qquad \eta_i(j) := \xi_{i/N^2}(j/N) \qquad (i \geqslant 0,\ 0 \leqslant j \leqslant N)$$

should satisfy the recursion

$$U_{i+1}(j) = \left(\frac{U_i(j+1) + U_i(j-1)}{2} + \lambda N U_i(j)\eta_i(j)\right)_+,$$

for all $0 \leqslant i \leqslant N^2$ and $1 \leqslant j \leqslant N-1$, where $a_+ := \max(a\,, 0)$. And

$$U_0(j) := U_{i+1}(0) := U_{i+1}(N) := 1 \qquad (i \geqslant 0,\ 0 \leqslant j \leqslant N).$$

It turns out that the $\eta_i(j)$'s are independent mean-zero normally-distributed random variables with common variance $\Delta t \Delta x = N^{-3}$. Therefore, $Z_i(j) := N^{3/2}\eta_i(j)$ defines a family of independent standard normal random variables. In this way we obtain the following:

$$U_{i+1}(j) = \left(\frac{U_i(j+1) + U_i(j-1)}{2} + \frac{\lambda U_i(j) Z_i(j)}{\sqrt{N}}\right)_+;$$

for all $i \geqslant 0$ and $1 \leqslant j \leqslant N-1$; $U_0(j) := \sin(\pi j/N)$; and $U_{i+1}(0) := U_{i+1}(N) := 0$ when $0 \leqslant j \leqslant N$. Therefore, if we wish to simulate $x \mapsto u_t(x)$ for every t up to a given time $T > 0$, then we choose a large mesh N, and then simulate—as above—$U_i(j)$ for every $0 \leqslant j \leqslant N$, where $i := \lfloor TN^2 \rfloor$.

[3]This is a "one-step Euler method," and can be shown to work reasonably well when $\Delta t, \Delta x \approx 0$. For this, and a great deal more, see the recent paper [42] by Walsh, for example.

A Matlab code. Here is the code that I used as a basis for the preceding simulations. It has not been optimized. Feel free to use it if you want. But be warned that the error of approximation becomes rapidly too big, as λ grows [$\lambda > 13$ is almost certainly too big for this method].

```
%------------------------------------------------------------%
FUNCTION      [MH] = heat (T,N,L)
%------------------------------------------------------------%
%   PARAMETERS:  T = terminal real time;                     %
%                N = the mesh size for the space variable;   %
%                L = coefficient of the noise (lambda);      %
%   OUTPUT:     MH = maximum height of the solution.         %
%------------------------------------------------------------%
tlim = floor(T*N^2);   % terminal simulation;
z = randn(tlim,N+1);   % standard normals are  generated;
u(1,:) = abs(sin(pi*(1:N+1)/(N+1)));   % initial data u(1,.);
FOR i=2:tlim
   u(i,1) = 0;
   u(i,N+1) = 0;                       % boundary values = 1;
   FOR j=2:N
      u(i,j) = max(0,(u(i-1,j+1)+u(i-1,j-1))/2+L*N^(-1/2)*u(i-1,j)*z(i,j));
   END
END
%------------------------------------------------------------%
MH = max(max(u)); %   Maximum height at terminal time T.
%------------------------------------------------------------%
```

Acknowledgements. These notes are the backbone of a five-lecture course on stochastic partial differential equations, given in Recife, Brazil in August 2012 as part of the *16th Brazilian School of Probability*. I am grateful to all of the scientific committee and the organizers of the summer school, most particularly to Professor ANDREI TOOM, for inviting me to this physically-spectacular, and intellectually-stimulating environment.

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1 Lecture 1: Itô and Wiener integrals

In this lecture we develop space-time white noise, also known as white noise in dimension 1+1. But first, I describe the construction of white noise in dimension one, since this is a slightly simpler object than space-time white noise, and is familiar to the audience of these lectures [albeit in slightly different form]. After that we study space-time white noise properly.

1.1 White noise in dimension 1. Let $\{W_t\}_{t\geqslant 0}$ denote a one-dimensional Brownian motion on a suitable probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and define

$$\dot{W}_t := \mathrm{d}W_t/\mathrm{d}t \qquad \text{for } t \geqslant 0.$$

The stochastic process $\{\dot{W}_t\}_{t\geqslant 0}$ is called *white noise* on $\mathbf{R}_+$, or *white noise in dimension 1*. Because W is nowhere differentiable with probability one [theorem of Paley et al [37]], we cannot define $\dot{W}$ as a classical stochastic process, rather only as a generalized random process, or better still a generalized random field [25]. One does this rigorously by mimicking distribution theory: For all $\phi \in C_c^\infty(\mathbf{R}_+)$—the space of infinitely-differentiable functions on $\mathbf{R}_+ := [0\,,\infty)$ with compact support—define

$$\dot{W}(\phi) := \int_0^\infty \phi_t \dot{W}_t \,\mathrm{d}t := -\int_0^\infty \dot{\phi}_t W_t \,\mathrm{d}t.$$

The right-most term is a well-defined Riemann integral; therefore the first two quantities are now well defined. Moreover, $\{\dot{W}(\phi)\}_{\phi\in C_c^\infty(\mathbf{R}_+)}$ is a centered Gaussian process with the following properties:

1. $\dot{W}(\alpha\phi + \beta\psi) = \alpha\dot{W}(\phi) + \beta\dot{W}(\psi)$ for all $\alpha, \beta \in \mathbf{R}$, $\phi, \psi \in C_c^\infty(\mathbf{R}_+)$; and
2. For every $\phi, \psi \in C_c^\infty(\mathbf{R}_+)$,

$$\mathrm{E}\left(\dot{W}(\phi)\dot{W}(\psi)\right) = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s\; \min(s\,,t)\,\dot{\phi}_t\dot{\psi}_t = \int_0^\infty \phi_t\psi_t \,\mathrm{d}t.$$

Therefore, $\dot{W}$ is a linear isometric embedding of $C_c^\infty(\mathbf{R}_+)$ into $L^2(\Omega)$. By density, $\dot{W}$ can be extended uniquely to a linear isometric embedding of $L^2(\mathbf{R}_+)$ in $L^2(\Omega)$. If $\phi \in L^2(\mathbf{R}_+)$ then $\dot{W}(\phi) = \int_0^\infty \phi_t \dot{W}_t \,\mathrm{d}t$ is also called the *Wiener integral* of ϕ. Our notation is suggestive but uncommon; usually people write $\int_0^\infty \phi_t \,\mathrm{d}W_t$ in place of our more suggestive $\int_0^\infty \phi_t\, \dot{W}_t \,\mathrm{d}t$. But the latter notation serves our needs much much better, as it turns out.

We can apply, purely formally, Fubini's theorem to see that

$$\mathrm{E}\left(\int_0^\infty \phi_t \dot{W}_t \,\mathrm{d}t \cdot \int_0^\infty \psi_t \dot{W}_s \,\mathrm{d}s\right) = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}s\; \phi_t\psi_s \mathrm{E}\left(\dot{W}_t\dot{W}_s\right),$$

for every $\phi, \psi \in L^2(\mathbf{R}_+)$. Since the left-hand side is rigorously equal to $\int_0^\infty \phi_t\psi_t \,\mathrm{d}t$ we can "conclude," seemingly non-rigorously, that

$$\mathrm{E}\left(\dot{W}_t\dot{W}_s\right) = \delta_0(t-s).$$

This appears to be non rigorous since $t \mapsto \dot{W}_t$ is not a random function. But in fact it is possible to create a theory of generalized Gaussian random fields that makes this rigorous [25], though the "δ-function" $(s\,,t) \mapsto \delta_0(t-s)$ has to be interpreted as a Carathéodory–Hausdorff type version of the Lebesgue measure on the diagonal of $\mathbf{R}_+ \times \mathbf{R}_+$. In this way, we are justified in saying that $\{\dot{W}_t\}_{t\geqslant 0}$ is a generalized Gaussian random field with mean measure 0 and *covariance measure* $\delta_0(t-s)$. I will skip the details of this theory of generalized random fields because we will only need the formal notation and not the machinations of the theory itself.

1.2 The Itô integral in dimension 1. As before, let $\{\dot{W}_t\}_{t\geqslant 0}$ denote white noise on $\mathbf{R}_+$, and $\{W_t\}_{t\geqslant 0}$ the corresponding Brownian motion. Now we wish to integrate random functions against white noise.

For all $t \geqslant 0$ define $\mathcal{F}_t$ to be the sigma-algebra generated by $\{W_s\}_{s\leqslant t}$. One can check that $\mathcal{F}_t$ can also be defined as the sigma-algebra generated by all variables for the form $\dot{W}(\mathbf{1}_{[0,t]}f) = \int_0^t f_r \dot{W}_r\,\mathrm{d}r$, as f ranges over $L^2(\mathbf{R}_+)$.

Suppose ϕ is an *elementary process*; i.e., has the form

$$\phi_t := X \cdot \mathbf{1}_{(\alpha,\beta)}(t),$$

where $0 \leqslant \alpha \leqslant \beta$ are non random, and $X \in L^2(\Omega)$ is $\mathcal{F}_\alpha$-measurable. Then we define

$$\dot{W}(\phi) := \int_0^\infty \phi_s \dot{W}_s\,\mathrm{d}s := X \cdot \int_\alpha^\beta \dot{W}_s\,\mathrm{d}s := X \cdot (W_\beta - W_\alpha).$$

The right-most term is well defined; therefore, the other three quantities are also well defined. Because X is independent of $\int_\alpha^\beta \dot{W}_s\,\mathrm{d}s = W_\beta - W_\alpha$, it follows also that

$$\mathrm{E}\dot{W}(\phi) = 0, \qquad \mathrm{E}\left(|\dot{W}(\phi)|^2\right) = \mathrm{E}(X^2)(\beta-\alpha) = \mathrm{E}\left(\|\phi\|^2_{L^2(\mathbf{R}_+)}\right).$$

Let $L^2_{\mathcal{P}}(\mathbf{R}_+)$ define the linear span, in $L^2(\Omega)$, of all elementary processes $\{\phi_t\}_{t\geqslant 0}$ of the preceding form. Elements of $L^2_{\mathcal{P}}(\mathbf{R}_+)$ are square-integrable *predictable* processes; this predictability accounts for the subscript $\mathcal{P}$.

Since $\phi \mapsto \dot{W}(\phi)$ is linear, the preceding defines a linear map $\dot{W}$ mapping $L^2_{\mathcal{P}}(\mathbf{R}_+)$ isometrically into $L^2(\Omega)$ such that $\mathrm{E}\dot{W}(\phi) = 0$ for all $\phi \in L^2_{\mathcal{P}}(\mathbf{R}_+)$. We continue to write

$$\dot{W}(\phi) := \int_0^\infty \phi_t \dot{W}_t\,\mathrm{d}t,$$

and refer to the right-hand side as the *Itô integral of* ϕ. In probability theory, we usually write $\int_0^\infty \phi_t\,\mathrm{d}W_t$ instead. But I will not do that in these lectures.

The space $L^2_{\mathcal{P}}(\mathbf{R}_+)$ is quite large; for example it contains all continuous stochastic processes $\{\phi_t\}_{t\geqslant 0}$ that are adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$ and satisfy $\mathrm{E}(\|\phi\|^2_{L^2(\mathbf{R}_+)}) < \infty$.

Proposition 1.1 (Itô, 1944). *$t \mapsto \int_0^t \phi_s \dot{W}_s\,\mathrm{d}s$ is a mean-zero continuous martingale with quadratic variation $\int_0^t \phi_s^2\,\mathrm{d}s$.*

Here is an outline of the proof of this proposition: First we prove it in the case that $\phi_t = X \cdot \mathbf{1}_{(\alpha,\beta)}(t)$ for an $\mathcal{F}_\alpha$-measurable $X \in L^2(\Omega)$. In that case, $\int_0^t \phi_s \dot{W}_s \, ds = X \cdot (W_{\beta\wedge t} - W_{\alpha\wedge t})$ clearly has the desired martingale properties. Then, we find $\phi^n \to \phi \in L^2_{\mathcal{P}}(\mathbf{R}_+)$ such that each ϕ^n is a finite nonrandom linear combination of terms of the form $X^n \cdot \mathbf{1}_{(\alpha_n,\beta_n)}(t)$ for $\mathcal{F}_{\alpha_n}$-measurable $X^n \in L^2(\Omega)$, where the intervals $(\alpha_n\,,\beta_n)$ are disjoint as n varies. A direct computation shows the isometry identity $\mathrm{E}(|\dot{W}(1_{(0,t)}\phi^n) - \dot{W}(1_{(0,t)}\phi^m)|^2) = \|(\phi^n - \phi^m)1_{(0,t)}\|^2_{L^2(\mathbf{R}_+)}$. Since $L^2(\Omega)$-limits of martingales are themselves $L^2(\Omega)$-martingales it follows that $M_t := \dot{W}(\mathbf{1}_{(0,t)}\phi)$ defines a mean-zero $L^2(\Omega)$-martingale. If we showed that M is a.s. continuous, then its quadratic variation at time t would also be $\int_0^t |\phi_s|^2 \, ds$ per force. The remaining continuity assertion follows from Doob's maximal inequality:

$$\mathrm{E}\left(\sup_{t\in(0,T)} |\dot{W}(\mathbf{1}_{(0,t)}\phi^n) - \dot{W}(\mathbf{1}_{(0,t)}\phi^m)|^2\right) \leqslant 4\|(\phi^n - \phi^m)1_{(0,T)}\|^2_{L^2(\mathbf{R}_+)},$$

valid as long as we used a standard augmentation of the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$.

1.3 White noise in dimension (1+1). White noise $\{\dot{W}_t(x)\}_{t\geqslant 0, x\in\mathbf{R}}$ in dimension (1+1) is the generalized Gaussian random field such that

$$\mathrm{E}\dot{W}_t(x) = 0 \quad \text{and} \quad \mathrm{E}\left(\dot{W}_t(x)\dot{W}_s(y)\right) = \delta_0(t-s)\delta_0(x-y),$$

where the product of the delta functions is understood rigorously as a measure on $(\mathbf{R}_+ \times \mathbf{R})^2$. More commonly, one refers to $\dot{W}$ as "space-time white noise."

In order to understand this object probabilistically, let us introduce a "two-sided Brownian sheet" $\{W_t(x)\}_{t\geqslant 0, x\in\mathbf{R}}$ as follows: $\{W_t(x)\}_{t\geqslant 0, x\in\mathbf{R}}$ is a centered Gaussian process with

$$\mathrm{E}\left(W_t(x)W_s(y)\right) = \min(s\,,t)\min(|x|\,,|y|)\mathbf{1}_{(0,\infty)}(xy).$$

We can check the following by directly computing the covariances:

1. For every fixed $t \geqslant 0$, $\{t^{-1/2}W_t(x)\}_{x\geqslant 0}$ and $\{t^{-1/2}W_t(-x)\}_{x\leqslant 0}$ are two independent Brownian motions [provided that 0/0 := 0]; and
2. For every $x \in \mathbf{R}$ fixed, $\{|x|^{-1/2}W_t(x)\}_{t\geqslant 0}$ defines a Brownian motion [again if 0/0 := 0].

Now we define $\dot{W}_t(x) := \partial^2_{tx} W_t(x)$, or more precisely the Wiener integral

$$\dot{W}(\phi) := \int_{\mathbf{R}_+\times\mathbf{R}} \partial^2_{tx}\phi_t(x) W_t(x) \, dt \, dx \qquad \text{for all } \phi \in C_c^\infty(\mathbf{R}_+ \times \mathbf{R}).$$

Note that $\{\dot W(\phi)\}_{\phi\in C_c^\infty(\mathbf{R}_+\times\mathbf{R})}$ defines a centered Gaussian random field with

$$\begin{aligned}
&\mathrm{E}\left(\dot W(\phi)\dot W(\psi)\right)\\
&=\int_{\mathbf{R}_+\times\mathbf{R}}\mathrm{d}t\,\mathrm{d}x\int_{\mathbf{R}_+\times\mathbf{R}}\mathrm{d}s\,\mathrm{d}y\ \frac{\partial^2\phi_t(x)}{\partial t\,\partial x}\frac{\partial^2\psi_s(y)}{\partial s\,\partial y}\min(s\,,t)\min(|x|\,,|y|)\mathbf{1}_{(0,\infty)}(xy)\\
&=\int_{\mathbf{R}_+\times\mathbf{R}}\phi_t(x)\psi_t(x)\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}$$

as one can check by considering functions $\phi_t(x)$ and $\psi_t(x)$ of product form $f_t\times g(x)$. This discussion implies that the Gaussian random field $\phi\mapsto\dot W(\phi)$ is a linear isometry from $C_c^\infty(\mathbf{R}_+\times\mathbf{R})$ into $L^2(\Omega)$. Therefore, we find by density a centered Gaussian random field $\{\dot W(\phi)\}_{\phi\in L^2(\mathbf{R}_+\times\mathbf{R})}$ such that $\phi\mapsto\dot W(\phi)$ is a linear isometry from $L^2(\mathbf{R}_+\times\mathbf{R})$ into $L^2(\Omega)$. We call $\dot W(\phi)$ the *Wiener integral* of $\phi\in L^2(\mathbf{R}_+\times\mathbf{R})$, and use alternatively the notation

$$\int_{\mathbf{R}_+\times\mathbf{R}}\phi_s(x)\dot W_s(x)\,\mathrm{d}s\,\mathrm{d}x\qquad\text{for all }\phi\in L^2(\mathbf{R}_+\times\mathbf{R}).$$

In this way we can also define consistently definite Wiener integrals

$$\int_{(a,b)\times\mathbf{R}}\phi_s(x)\dot W_s(x)\,\mathrm{d}s\,\mathrm{d}x:=\dot W\left(\mathbf{1}_{(a,b)}\phi\right),$$

for all $0<a<b$ and $\phi\in L^2(\mathbf{R}_+\times\mathbf{R})$. Note, in particular, that $(t\,,\phi)\mapsto\int_{(0,t)\times\mathbf{R}}\phi_s(x)\dot W_s(x)\,\mathrm{d}s\,\mathrm{d}x$ defines a Gaussian random field with mean zero and covariance

$$\begin{aligned}
\mathrm{Cov}\left(\int_{(0,t)\times\mathbf{R}}\phi_s(x)\dot W_s(x)\,\mathrm{d}s\,\mathrm{d}x\,,\int_{(0,t')\times\mathbf{R}}\psi_s(x)\dot W_s(x)\,\mathrm{d}s\,\mathrm{d}x\right)&\\
=\int_0^{t\wedge t'}\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}x\ \phi_s(x)\psi_s(x),&
\end{aligned}\tag{1.1}$$

for all $0<t<t'$ and $\phi,\psi\in L^2(\mathbf{R}_+\times\mathbf{R}_+)$.

2 Lecture 2: Stochastic convolutions

2.1 Walsh integrals. Let $\mathcal{F}_t$ define the sigma-algebra generated by the collection of random variables $\{W_s(x)\}_{s\in(0,t),x\in\mathbf{R}}$. Equivalently, $\mathcal{F}_t$ denotes the sigma-algebra generated by $\{\dot{W}(\mathbf{1}_{(0,t)}\phi)\}_{\phi\in L^2(\mathbf{R}_+\times\mathbf{R})}$.[4]

Consider a random field $\phi = \{\phi_t(x)\}_{t\geqslant 0,x\in\mathbf{R}}$ of the form

$$\phi_t(x) = \mathbf{1}_{(\alpha,\beta)}(t)\; X\cdot f(x),$$

where $0 \leqslant \alpha \leqslant \beta$, $X \in L^2(\Omega)$ is $\mathcal{F}_\alpha$-measurable, and $f \in L^2(\mathbf{R})$ is uniformly bounded. Such random fields are called *elementary*. We can define, for all elementary ϕ,

$$\dot{W}(\phi) := \int_{\mathbf{R}_+\times\mathbf{R}} \phi_t(x)\dot{W}_t(x)\,\mathrm{d}t\,\mathrm{d}x := X\cdot \dot{W}\left(\mathbf{1}_{(\alpha,\beta)}f\right).$$

Since the right-most quantity is well defined as a Wiener integral, the preceding defines the first two terms rigorously. We may observe that $\dot{W}(\mathbf{1}_{(\alpha,\beta}f)$ is independent of $\dot{W}(\mathbf{1}_{(0,\alpha)}g)$ for all $g \in L^2(\mathbf{R})$ since the correlation between the two Wiener integrals is the inner product—in $L^2(\mathbf{R}_+\times\mathbf{R})$—between $\mathbf{1}_{(0,\alpha)}g$ and $\mathbf{1}_{(\alpha,\beta)}f$. Therefore, X is independent of $\dot{W}(\mathbf{1}_{(\alpha,\beta)}f)$, whence

$$\mathrm{E}\dot{W}(\phi) = 0, \quad \mathrm{E}\left(|\dot{W}(\phi)|^2\right) = \mathrm{E}(X^2)\left\|\mathbf{1}_{(\alpha,\beta)}f\right\|^2_{L^2(\mathbf{R}_+\times\mathbf{R})} = \mathrm{E}\left(\|\phi\|^2_{L^2(\mathbf{R}_+\times\mathbf{R})}\right).$$

Finite linear combinations of simple random fields are called *simple*. We can define $\dot{W}(\phi)$ for a simple ϕ by linearity. Let $L^2_{\mathcal{P}}(\mathbf{R}_+\times\mathbf{R})$ denote the linear span of all such random fields ϕ in $L^2(\Omega\times\mathbf{R}_+\times\mathbf{R})$. Elements of $L^2_{\mathcal{P}}(\mathbf{R}_+\times\mathbf{R})$ are called *predictable random fields*. The preceding defines uniquely a linear isometric embedding $\dot{W}$ of $L^2_{\mathcal{P}}(\mathbf{R}_+\times\mathbf{R})$ into $L^2(\Omega)$ such that

$$\mathrm{E}\dot{W}(\phi) = 0 \quad\text{and}\quad \mathrm{E}\left(|\dot{W}(\phi)|^2\right) = \mathrm{E}\left(\|\phi\|^2_{L^2(\mathbf{R}_+\times\mathbf{R})}\right).$$

I will write

$$\dot{W}(\phi) := \int_{\mathbf{R}_+\times\mathbf{R}} \phi_t(x)\dot{W}_t(x)\,\mathrm{d}t\,\mathrm{d}x, \qquad \text{for all } \phi\in L^2_{\mathcal{P}}(\mathbf{R}_+\times\mathbf{R}),$$

and refer to $\dot{W}(\phi)$ as the *Walsh integral* of ϕ. In the probability literature, we usually write $\int_0^\infty\int_{-\infty}^\infty \phi_t(x)\,W(\mathrm{d}t\,\mathrm{d}x)$ instead; but I prefer the displayed notation and will not do that in these lectures.

The space $L^2_{\mathcal{P}}(\mathbf{R}_+\times\mathbf{R})$ is very large; for instance it contains all random processes of the form $\phi_t(x) = f(t\,,W_t(x))$ where $f:\mathbf{R}_+\times\mathbf{R}\to\mathbf{R}$ is lower semicontinuous and satisfies $\mathrm{E}(\int_0^\infty \mathrm{d}t\int_{-\infty}^\infty \mathrm{d}x\,|f(t\,,W_t(x))|^2) < \infty$.

[4]We will be tacitly replace $\mathcal{F}_t$ by its usual augmentation, as described for example in Dellacherie and Meyer [19].

2.2 A connection to martingales. If $\phi \in L^2(\mathbf{R})$, then

$$M_t := \int_{(0,t)\times\mathbf{R}} \phi(y)\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y := \dot{W}\left(\mathbf{1}_{(0,t)}\phi\right) \qquad (t>0)$$

defines a Gaussian process with mean zero. In fact, whenever $s,t \geqslant 0$,

$$\mathrm{E}\left(M_t M_{t+s}\right) = \int_{\mathbf{R}_+\times\mathbf{R}} \mathbf{1}_{(0,t)}(r)\phi(x)\mathbf{1}_{(0,t+s)}(r)\phi(x)\,\mathrm{d}r\,\mathrm{d}x = t\|\phi\|^2_{L^2(\mathbf{R})}.$$

Therefore, $t \mapsto M_t/\|\phi\|_{L^2(\mathbf{R})}$ defines a Brownian motion. This and a density argument together yield the following.

Proposition 2.1 (Walsh [43]). *For every $\phi \in L^2_{\mathcal{P}}(\mathbf{R}_+\times\mathbf{R})$ and $t \geqslant 0$ define*

$$M_t(\phi) := \int_{(0,t)\times\mathbf{R}} \dot{W}_s(y)\phi_s(y)\,\mathrm{d}s\,\mathrm{d}y := \dot{W}\left(\mathbf{1}_{(0,t)}\phi\right).$$

Then $\{M_t(\phi)\}_{t\geqslant 0}$ is a continuous mean-zero $L^2(\Omega)$-martingale with quadratic variation

$$\langle M(\phi)\rangle_t = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ |\phi_s(y)|^2 = \left\|\mathbf{1}_{(0,t)}\phi\right\|^2_{L^2(\mathbf{R}_+\times\mathbf{R})}.$$

The following consequence is of paramount importance to us.

Corollary 2.2 (The BDG inequality). *Choose and fix $k \in [2,\infty)$. If M is a continuous martingale with $M_t \in L^k(\Omega)$ for all $t \geqslant 0$, then*

$$\mathrm{E}\left(|M_t|^k\right) \leqslant (4k)^{k/2}\mathrm{E}\left(|\langle M\rangle_t|^{k/2}\right) \qquad (t \geqslant 0),$$

where $\langle M\rangle_t$ denotes the quadratic variation of M.

This is the usual BDG inequality, due to Burkholder, Davis, and Gundy, for continuous $L^2(\Omega)$-martingales [5–7]. But we have also used the fact that the best constant in that inequality is at most $(4k)^{k/2}$; see the bound by Carlen and Kree [9] on the optimal constant in the BDG inequality, found earlier by Davis [18].

An application of the preceding together with the Minkowski inequality yields the following. From now on, I will write $\|\cdots\|_k$ in place of $\|\cdots\|_{L^k(\Omega)}$. That is,

$$\|Z\|_k := \left\{\mathrm{E}\left(|Z|^k\right)\right\}^{1/k}, \tag{2.1}$$

for every random variable $Z \in L^k(\Omega)$, and $k \in [1,\infty)$.

Corollary 2.3 (The BDG inequality [22]). *If $\phi \in L^2_{\mathcal{P}}(\mathbf{R}_+\times\mathbf{R})$, then for all $k \geqslant 2$,*

$$\left\|\int_{\mathbf{R}_+\times\mathbf{R}} \phi_s(x)\dot{W}_s(x)\,\mathrm{d}s\,\mathrm{d}x\right\|_k^2 \leqslant 4k\int_0^\infty \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}x\ \|\phi_s(x)\|_k^2.$$

2.3 Stochastic convolutions. One can define quite general "stochastic convolutions." However, we define only what we need. With this in mind, let us first denote by $p_t(x)$ the following normalization of the heat kernel on $\mathbf{R}$:

$$p_t(x) := \frac{\mathrm{e}^{-x^2/(2t)}}{(2\pi t)^{1/2}} \qquad (t > 0,\, x \in \mathbf{R}).$$

If Z is a predictable random field that satisfies

$$\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{t-s}(y-x)]^2 \mathrm{E}\left(|Z_s(y)|^2\right) < \infty \qquad (t > 0,\, x \in \mathbf{R}), \tag{2.2}$$

then we let

$$M_{t,\tau}(x) := \int_{(0,t)\times\mathbf{R}} p_{\tau-s}(y-x) Z_s(y) \dot{W}_s(y)\, \mathrm{d}s\, \mathrm{d}y \qquad (0 < t < \tau,\, x \in \mathbf{R}).$$

The stochastic integral is defined in the sense of Walsh, and $\{M_{t,\tau}(x)\}_{t\in[0,\tau]}$ is a continuous mean-zero martingale for every fixed $\tau > 0$. The *stochastic convolution*

$$\left(p * Z\dot{W}\right)_t(x) := M_{t,t}(x) = \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x) Z_s(y)\, \dot{W}_s(y)\, \mathrm{d}s\, \mathrm{d}y$$

is therefore a well-defined mean-zero random field, whose variance is described by the left-hand side of (2.2).

Define for all $k \in [2\,,\infty)$, $\beta > 0$, and every predictable random field Z,

$$\mathcal{N}_\beta^k(Z) := \sup_{t>0} \sup_{x\in\mathbf{R}} \left(\mathrm{e}^{-\beta t} \|Z_t(x)\|_k\right). \tag{2.3}$$

It is not hard to see that if Z is a predictable random field that satisfies $\mathcal{N}_\beta^2(Z) < \infty$ for some $\beta > 0$ and $k \in [2\,,\infty)$ then Z also satisfies (2.2), and hence $p*Z\dot{W}$ is a well-defined square-integrable random field. The following is a stronger statement.

Proposition 2.4 (Conus, Foondun, and K [14, 22])**.** *For all $k \in [2\,,\infty)$, $\beta > 0$, and every predictable random field Z,*

$$\mathcal{N}_\beta^k\left(p * Z\dot{W}\right) \leqslant \frac{(2k)^{1/2}}{\beta^{1/4}} \mathcal{N}_\beta^k(Z).$$

Proof of Proposition 2.4. We apply the BDG inequality, in the form mentioned earlier (Corollary 2.3), and deduce that

$$\begin{aligned}
&\mathrm{e}^{-2\beta t} \left\| \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x) Z_s(y) \dot{W}_s(y)\, \mathrm{d}s\, \mathrm{d}y \right\|_k^2 \\
&\qquad\leqslant 4k\mathrm{e}^{-2\beta t} \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{t-s}(y-x)]^2 \|Z_s(y)\|_k^2 \\
&\qquad\leqslant 4k \left[\mathcal{N}_\beta^k(Z)\right]^2 \cdot \int_0^t \mathrm{e}^{-2\beta(t-s)} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{t-s}(y-x)]^2 \\
&\qquad\leqslant 4k \left[\mathcal{N}_\beta^k(Z)\right]^2 \cdot \int_0^\infty \mathrm{e}^{-2\beta s} \|p_s\|_{L^2(\mathbf{R})}^2\, \mathrm{d}s.
\end{aligned}$$

In particular, we may optimize the left-most term over all x and t, in order to see that

$$\left[\mathcal{N}_\beta^k\left(p * Z\dot{W}\right)\right]^2 \leqslant 4k\left[\mathcal{N}_\beta^k(Z)\right]^2 \cdot \int_0^\infty \mathrm{e}^{-2\beta s}\|p_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s.$$

A direct computation shows that the preceding integral is equal to $(4\beta)^{-1/2}$, and the proposition follows. □

Now define $\mathbf{L}_\beta^k$ to be the completion, in the norm $\mathcal{N}_\beta^k$, of the vector space all predictable random fields Z such that $\mathcal{N}_\beta^k(Z) < \infty$.

Corollary 2.5. *Choose and fix $\beta > 0$ and $k \in [2\,,\infty)$. Then, the stochastic convolution map $Z \mapsto p * Z\dot{W}$ defines a bounded linear operator from $\mathbf{L}_\beta^k$ into itself, with operator norm being no more than $(2k)^{1/2}\beta^{-1/4}$. Moreover, $(x\,,t) \mapsto (p * Z\dot{W})_t(x)$ has a continuous modification.*

Proof. The continuity assertion will follow from the ensuing remarks. If so then, in light of the preceding proposition, it suffices to prove that if $Z \in \mathbf{L}_\beta^k$ is predictable then $p * Z\dot{W}$ is predictable also. In fact it is enough to prove this assertion in the case that Z is an elementary function. But then our problem is reduced to the case that $Z_s(y) = \mathbf{1}_{[\alpha,\beta]}(t)f(x)$ for a nonrandom and bounded $f \in L^2(\mathbf{R})$. In this case, a few elementary estimates show that the Gaussian random field $p * Z\dot{W}$ satisfies

$$\left\|\left(p * Z\dot{W}\right)_t(x) - \left(p * Z\dot{W}\right)_t(x')\right\|_2 = O\left(|x - x'|^{1/2}\right),$$

and

$$\left\|\left(p * Z\dot{W}\right)_t(x) - \left(p * Z\dot{W}\right)_{t'}(x)\right\|_2 = O\left(|t - t'|^{1/4}\right),$$

uniformly for all $x, x' \in \mathbf{R}$ and $t, t' \in [0\,,T]$, where $T > 0$ is fixed but arbitrary; see [17, Ch. 1]. It follows readily from this that $[0\,,T] \times \mathbf{R} \ni (p * Z\dot{W})_t(x)$ is continuous in $\cap_{p\in[2,\infty)}L^p(\mathrm{P})$ (and also almost surely, thanks to a suitable form of KOLMOGOROV's continuity theorem) and therefore a predictable random field. □

3 Lecture 3: A stochastic heat equation

3.1 Existence and uniqueness. Let $\sigma : \mathbf{R} \to \mathbf{R}$ be a globally Lipschitz function; recall that this means that

$$\mathrm{Lip}_\sigma := \sup_{x \neq y} \frac{|\sigma(x) - \sigma(y)|}{|x - y|} < \infty.$$

Our goal is to solve the following

$$\left[\begin{array}{l} \partial_t u_t(x) = \frac{1}{2}\partial^2_{xx} u_t(x) + \sigma(u_t(x))\dot{W}_t(x), \\ \text{subject to } u_0 \text{ being bounded, measurable, and nonrandom.} \end{array}\right. \tag{3.1}$$

If $\dot{W}_t(x)$ were replaced by a smooth function, then classical PDEs tells us that the solution to (3.1) is[5]

$$u_t(x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y. \tag{3.2}$$

In the stochastic setting, our notion of solution remains the same, but we interpret the integral that involves the noise $\dot{W}$ as a stochastic convolution. It can be proved that the resulting "solution," when it exists, is a "weak solution" to (3.1). Any solution to (3.2) is called a *mild solution* to (3.1).

Theorem 3.1 (Dalang [16], Foondun and K [22], Walsh [43])**.** *The stochastic heat equation* (3.1) *has a mild solution u that is unique within $\mathbf{L}^k_\beta$ for all $\beta > 0$ and $k \in [2, \infty)$. In addition, there exists a universal constant $C \in (0, \infty)$ such that for all $k \geqslant 2$ and $t > 0$,*

$$\sup_{x\in\mathbf{R}} \mathrm{E}\left(|u_t(x)|^k\right) \leqslant C \exp\left(Ck^3 t\right).$$

Remark 3.2. This moment condition is sharp. For example, if, in addition, $\inf_{x\in\mathbf{R}} |\sigma(x)/x| > 0$ and $\inf_{x\in\mathbf{R}} u_0(x) > 0$, then one can prove that there exists $c \in (0, 1)$ such that $\inf_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^k) \geqslant c\exp(ck^3 t)$ for all $k \geqslant 2$ and $t > 0$, as well; see Conus et al [12]. Since the kth moment grows very rapidly with k, this suggests strongly that the distribution of $u_t(x)$ might be not determined by its moments; this is further corroborated by the somewhat exotic form of the tail estimate (5.3) below. □

Proof (sketch). For the most part we follow the well-known Picard iteraton method from classical ODEs. Therefore, I will concentrate mostly on the novel features of the proof and ask you to fill in the mostly-standard details.

Define $u^{(0)}_t(x) := u_0(x)$, and for all $n \geqslant 0$, $t > 0$, and $x \in \mathbf{R}$,

$$u^{(n+1)}_t(x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma\left(u^{(n)}_s(y)\right)\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y.$$

[5] This method is also known as the method of variation of constants, and the resulting mild formulation of the solution is called Duhamel's formula.

Because

$$u^{(n+1)} - u^{(n)} = p * \left[\sigma\left(u^{(n)}\right) - \sigma\left(u^{(n-1)}\right)\right] \dot{W},$$

it follows from Proposition 2.4 (p. 13) that for all $\beta > 0$ and $k \in [2\,,\infty)$,

$$\begin{aligned}\mathcal{N}_\beta^k\left(u^{(n+1)} - u^{(n)}\right) &\leqslant \frac{(2k)^{1/2}}{\beta^{1/4}} \mathcal{N}_\beta^k\left(\sigma\left(u^{(n)}\right) - \sigma\left(u^{(n-1)}\right)\right) \\ &\leqslant \frac{(2k)^{1/2}}{\beta^{1/4}} \mathrm{Lip}_\sigma\, \mathcal{N}_\beta^k\left(u^{(n)} - u^{(n-1)}\right),\end{aligned}$$

where, we recall, Lip_σ denotes the Lipschitz constant of σ. We can now apply the preceding, together with a "fixed-point argument," to deduce that if $u^{(n)} \in \mathbf{L}_\beta^k$ for all n, and if also

$$\beta > (2k)^2 \mathrm{Lip}_\sigma^4, \tag{3.3}$$

then $u := \lim_{n\to\infty} u^{(n)}$ exists in $\mathbf{L}_\beta^k$, and solves our stochastic PDE. A similar inequality shows also that $\sup_n \mathcal{N}_\beta^k(u^{(n)}) < \infty$ provided that (3.3) holds. All this has the desired result, and the asserted moment estimate follows because $\mathcal{N}_\beta^k(u) < \infty$ for any β that satisfies (3.3) [check the arithmetic!]. $\square$

3.2 Lyapunov exponents. Choose and fix some $x \in \mathbf{R}$, and define for all real numbers $k \geqslant 0$,

$$\overline{\gamma}(k) := \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}\left(|u_t(x)|^k\right).$$

The quantity $\overline{\gamma}(k)$ is the *upper kth moment Lyapounov exponent* of the solution at x, and Theorem 3.1 implies that $\overline{\gamma}(k)$ is finite for all $k \geqslant 0$. By Jensen's inequality, if $1 \leqslant k \leqslant \ell < \infty$ then $\|u_t(x)\|_k \leqslant \|u_t(x)\|_\ell$, whence

$$\frac{1}{kt} \log \mathrm{E}\left(|u_t(x)|^k\right) \leqslant \frac{1}{\ell t} \log \mathrm{E}\left(|u_t(x)|^\ell\right).$$

We first let $t \to \infty$, and then $k, \ell \to \infty$ in different ways to find that

$$\frac{\overline{\gamma}(k)}{k} \text{ is nondecreasing for } k \in [1\,,\infty).$$

The following is due to Carmona and Molchanov [8, p. 55].

Lemma 3.3. *Suppose $u_t(x) \geqslant 0$ a.s. for all $t \geqslant 0$ and $x \in \mathbf{R}$, $\overline{\gamma}(k) < \infty$ for all $k < \infty$, and $\overline{\gamma}(c) > 0$ for some $c > 1$. Then, $\overline{\gamma}(k)/k$ is strictly increasing for $k \geqslant c$.*

It has been proposed that we call the solution to (3.1) *intermittent* [or weakly intermittent] if $\overline{\gamma}(k)/k$ is strictly increasing. You can find in the introduction of Bertini and Cancrini's paper [4] a heuristic justification for why this mathematical property implies that the solution tends to develop large peaks [see, for example, the simulation of the slightly different SPDE (2) on page 3 of these lectures.] Moreover, the mentioned heuristic suggests strongly that the height of the tall peaks grow exponentially fast with time. In other words, the stochastic heat equation behaves increasingly differently from the linear heat equation as $t \to \infty$.

Enough said; let us prove something next.

Proof of Lemma 3.3. Because u is nonnegative,

$$\overline{\gamma}(k) = \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}\left(u_t(x)^k\right) \qquad \text{for all } k \geqslant 0. \tag{3.4}$$

[N.B.: No absolute values!] Since $\mathrm{E}u_t(x) = (p_t * u_0)(x)$ is bounded above uniformly by $\sup_x u_0(x)$ it follows that

$$\overline{\gamma}(1) = 0 < \overline{\gamma}(c). \tag{3.5}$$

Next we claim that $\overline{\gamma}$ is convex on $\mathbf{R}_+$. Indeed, for all $a, b \geqslant 0$ and $\lambda \in (0\,,1)$, HÖLDER's inequality yields the following: For all $p \in (1\,,\infty)$ with $q := p/(p-1)$,

$$\mathrm{E}\left[u_t(x)^{\lambda a+(1-\lambda)b}\right] \leqslant \left\{\mathrm{E}\left[u_t(x)^{p\lambda a}\right]\right\}^{1/p} \left\{\mathrm{E}\left[u_t(x)^{q(1-\lambda)b}\right]\right\}^{1/q}.$$

Choose $p := 1/\lambda$ to deduce the convexity of $\overline{\gamma}$ from (3.4).

Now we complete the proof: By (3.5) and convexity, $\overline{\gamma}(k) > 0$ for all $k \geqslant 2$. If $k' > k \geqslant c$, then we write $k = \lambda k' + (1-\lambda)$—with $\lambda := (k-1)/(k'-1)$—and apply convexity to conclude that

$$\overline{\gamma}(k) \leqslant \lambda\overline{\gamma}(k') + (1-\lambda)\overline{\gamma}(1) = \frac{k-1}{k'-1}\,\overline{\gamma}(k'). \tag{3.6}$$

Since (3.6) holds in particular with $k = c$, it implies that $\overline{\gamma}(k') > 0$. And the lemma follows from (3.6) and the inequality $(k-1)/(k'-1) < k/k'$. ☐

The following yields natural conditions under which the solution is indeed non negative. In PDEs, such results are proved by means of a maximum principle. Although SPDEs do not have a maximum principle, they fortunately *do* have a comparison principle.

Theorem 3.4 (MUELLER's comparison principle [36]). *If $\sigma(0) = 0$ and $u_0(x) \geqslant 0$ for all $x \in \mathbf{R}$, then $u_t(x) \geqslant 0$ a.s. for all $t > 0$ and $x \in \mathbf{R}$.*

As far as the intermittency of the solution is concerned, it remains to discover conditions under which $\overline{\gamma}(c) > 0$ for some $c > 1$. We will do this momentarily. However, let us make an aside before going further.

For every $c \in \mathbf{R}$,

$$\mathrm{P}\left\{u_t(x) \geqslant \mathrm{e}^{ct}\right\} \leqslant \mathrm{e}^{-ckt}\|u_t(x)\|_k^k \leqslant \exp\left\{-tk\left[c - (1+o(1))\frac{\overline{\gamma}(k)}{k}\right]\right\}.$$

Take logarithms and let $t \to \infty$ to see that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathrm{P}\left\{u_t(x) \geqslant \mathrm{e}^{ct}\right\} \leqslant -\sup_{k>0}\left[k\left(c - \frac{\overline{\gamma}(k)}{k}\right)\right] < 0 \text{ for all } c > 0.$$

In other words, we are confident [in the sense of large deviations] that the solution does not grow exponentially with time. This is consistent with properties of [nonrandom] heat equations.

Open Problem. It is not hard to show that $\lim_{k\downarrow 0}(\overline{\gamma}(k)/k)$ exists; the limit is believed to be strictly negative for a large family of nonlinearities σ. If so, then the preceding argument shows in fact that there exists $q = -c > 0$ such that with $u_t(x) < \mathrm{e}^{-qx}$ with overwhelming probability. There are now instances of concrete SPDEs where this has been proved; see for example AMIR ET AL [2]. It would be interesting if there were a way to prove this more generally. □

3.3 A lower bound. Recall that the remaining issue with intermittency for the solution to (3.1) is to verify that $\overline{\gamma}(c) > 0$ for some $c > 1$. The following does exactly that in some cases.

Theorem 3.5 (FOONDUN and K [22]). *We have* $\overline{\gamma}(2) > 0$*, provided that* $\inf_{x\in\mathbf{R}} u_0(x) > 0$ *and* $\inf_{z\in\mathbf{R}} |\sigma(z)/z| > 0$*.*

Proof. Let $I_0 := \inf_{x\in\mathbf{R}} u_0(x)$ and $J_0 := \inf_{z\in\mathbf{R}} |\sigma(z)/z|$. For all $x \in \mathbf{R}$ fixed,

$$\begin{aligned} \mathrm{E}\left(|u_t(x)|^2\right) &= |(p_t * u_0)(x)|^2 + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ p_{t-s}^2(y-x)\mathrm{E}\left(|\sigma(u_s(y))|^2\right) \\ &\geqslant I_0^2 + J_0^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ p_{t-s}^2(y-x)\mathrm{E}\left(|u_s(y)|^2\right). \end{aligned} \tag{3.7}$$

Therefore,

$$N_\beta(x) := \int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}\left(|u_t(x)|^2\right) \mathrm{d}t$$

satisfies

$$N_\beta(x) \geqslant \frac{I_0^2}{\beta} + J_0^2 \cdot \int_{-\infty}^{\infty} A_\beta(y-x) N_\beta(y)\, \mathrm{d}y,$$

where

$$A_\beta(z) := \int_0^\infty \mathrm{e}^{-\beta t} p_t^2(z)\, \mathrm{d}t.$$

Therefore,

$$\begin{aligned} \inf_{x\in\mathbf{R}} N_\beta(x) &\geqslant \frac{I_0^2}{\beta} + J_0^2 \cdot \inf_{x\in\mathbf{R}} N_\beta(x) \cdot \int_{-\infty}^{\infty} A_\beta(y)\, \mathrm{d}y \\ &= \frac{I_0^2}{\beta} + J_0^2 \cdot \inf_{x\in\mathbf{R}} N_\beta(x) \cdot \int_0^\infty \mathrm{e}^{-\beta t} \|p_t\|_{L^2(\mathbf{R})}^2\, \mathrm{d}t \\ &= \frac{I_0^2}{\beta} + \mathrm{const} \cdot \frac{J_0^2}{\sqrt{\beta}} \cdot \inf_{x\in\mathbf{R}} N_\beta(x). \end{aligned} \tag{3.8}$$

If $\beta > 0$ is sufficiently small, then the coefficient of $\inf_{x\in\mathbf{R}} N_\beta(x)$ on the right-most term is > 1. Because $I_0 > 0$, this implies that $\inf_{x\in\mathbf{R}} N_\beta(x) = \infty$ for such a β. That is,

$$N_\beta(x) = \int_0^\infty e^{-\beta t} \mathrm{E}\left(|u_t(x)|^2\right) \mathrm{d}t = \infty \quad \text{simultaneously for all } x \in \mathbf{R},$$

for β sufficiently small. This has the desired effect. Indeed, suppose to the contrary that $\overline{\gamma}(2) = \limsup_{t\to\infty} t^{-1} \log \mathrm{E}(|u_t(x)|^2) = 0$. Then, there exists t_0 large enough such that $\mathrm{E}(|u_t(x)|^2) \leqslant \exp(\beta t/2)$ for $t > t_0$, whence

$$\int_{t_0}^\infty e^{-\beta t} \mathrm{E}\left(|u_t(x)|^2\right) \mathrm{d}t \leqslant \int_{t_0}^\infty e^{-\beta t/2} \,\mathrm{d}t < \infty,$$

and this is a contradiction. □

The preceding has some variants that are interesting as well. Let me mention one such result without proof.

Theorem 3.6 (Foondun and K [22]). *Assume* $\liminf_{|z|\to\infty} |\sigma(z)/z| > 0$ *and* $I_0 := \inf_{x\in\mathbf{R}} u_0(x)$ *is strictly positive. Then* $\overline{\gamma}(2) > 0$ *provided that* $\inf_x u_0(x)$ *is large enough.*

4 Lecture 4: On compact-support initial data

4.1 The position of the high peaks. The condition that $\inf_{x\in\mathbf{R}} u_0(x) > 0$ is very strong. In this section we analyze the complimentary case where u_0 decays at infinity. At this time, we are only able to study the case of exponential decay, but our analysis has the added benefit that it tells us about the position of the peaks where intermittency occurs. Let us define two indices:

$$\overline{\lambda}(k) := \inf\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geqslant \alpha t} \log \mathrm{E}\left(|u_t(x)|^k\right) < 0\right\};$$

where $\inf\varnothing := \infty$; and

$$\underline{\lambda}(k) := \sup\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geqslant \alpha t} \log \mathrm{E}\left(|u_t(x)|^k\right) > 0\right\};$$

It is easy to see that

$$0 \leqslant \underline{\lambda}(k) \leqslant \overline{\lambda}(k) \leqslant \infty \qquad \text{for all } k \geqslant 2.$$

We are interested to know when the extreme inequalities are strict. In that case, one can make a heuristic argument that states that we have intermittency, and moreover the farthest high peaks travel, away from the origin, roughly at linear speed with time.

Theorem 4.1 (Conus and K [14]). *Suppose $u_0 : \mathbf{R} \to \mathbf{R}_+$ is lower semicontinuous, strictly positive on a set of strictly-positive measure, and satisfies $|u_0(x)| = O(\mathrm{e}^{-\rho|x|})$ as $|x| \to \infty$ for some $\rho > 0$. If, in addition, $\sigma(0) = 0$ and $\inf_{x\in\mathbf{R}} |\sigma(x)/x| > 0$, then*

$$0 < \underline{\lambda}(k) \leqslant \overline{\lambda}(k) < \infty \qquad \textit{for all } k \geqslant 2.$$

Conjecture. I believe that the middle inequality is an identity. Le Chen and Robert Dalang have recently verified this conjecture in the physically-important case that $\sigma(x) \propto x$ [that is the so-called parabolic Anderson model]. If this is so, then it implies that the farthest high peaks travel exactly at linear speed with time, away from the origin. Moreover, there is a phase separation when $\underline{\lambda}(k) = \overline{\lambda}(k) : -\lambda(k)$: If $|x| > \lambda(k)t(1+\epsilon)$, then there is almost no "mass" at x [for t large]; whereas there is exponentially-large mass at some $|x| \simeq \lambda(k)t(1 \pm o(1))$ when t is large.

4.2 Proof. We say that $\vartheta : \mathbf{R} \to \mathbf{R}_+$ is a *weight* when ϑ is measurable and

$$\vartheta(a+b) \leqslant \vartheta(a)\vartheta(b) \qquad \text{for all } a, b \in \mathbf{R}.$$

As usual, the weighted L^2-space $L^2_\vartheta(\mathbf{R})$ denotes the collection of all measurable functions $h : \mathbf{R} \to \mathbf{R}$ such that $\|h\|_{L^2_\vartheta(\mathbf{R})} < \infty$, where

$$\|h\|^2_{L^2_\vartheta(\mathbf{R})} := \int_{-\infty}^{\infty} |h(x)|^2\, \vartheta(x)\, \mathrm{d}x.$$

Define, for all predictable processes v, every real number $k \geqslant 1$, and all $\beta > 0$,

$$\mathcal{N}_{\beta,k,\vartheta}(v) := \left[\sup_{t\geqslant 0}\sup_{x\in\mathbf{R}} \mathrm{e}^{-\beta t}\vartheta(x)\,\|v_t(x)\|_k^2\right]^{1/2}.$$

Suppose $\Gamma_t(x)$ is a nonnegative, nonrandom, measurable function and $Z \in L^2_{\mathcal{P}}(\mathbf{R}_+ \times \mathbf{R})$. Let us write, for shorthand, $\Gamma \circledast Z\dot{W}$ for the random field

$$(\Gamma \circledast Z\dot{W})_t(x) := \int_{(0,t)\times\mathbf{R}} \Gamma_{t-s}(y-x) Z_s(y)\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y.$$

Proposition 4.2 (A stochastic Young inequality; Conus and K [14]). *For all weights ϑ, all $\beta > 0$, and all $k \geqslant 2$,*

$$\mathcal{N}_{\beta,k,\vartheta}(\Gamma \circledast Z\dot{W}) \leqslant \left(4k\int_0^\infty \mathrm{e}^{-\beta t}\,\|\Gamma_t\|^2_{L^2_\vartheta(\mathbf{R})}\,\mathrm{d}t\right)^{1/2} \cdot \mathcal{N}_{\beta,k,\vartheta}(Z).$$

Proof. We apply our corollary to the BDG inequality as follows:

$$\begin{aligned}
&\mathrm{e}^{-\beta t}\vartheta(x)\left\|(\Gamma \circledast Z\dot{W})_t(x)\right\|_k^2\\
&\leqslant 4k\int_{(0,t)\times\mathbf{R}} \mathrm{e}^{-\beta(t-s)}\vartheta(y-x)\Gamma^2_{t-s}(y-x)\,\mathrm{e}^{-\beta s}\vartheta(y)\,\|Z_s(y)\|_k^2\,\mathrm{d}s\,\mathrm{d}y \qquad (4.1)\\
&\leqslant 4k\,|\mathcal{N}_{\beta,k,\vartheta}(Z)|^2 \cdot \int_{(0,t)\times\mathbf{R}} \mathrm{e}^{-\beta r}\vartheta(z)\Gamma_r^2(z)\,\mathrm{d}r\,\mathrm{d}z.
\end{aligned}$$

The proposition follows from optimizing this expression over all $t \geqslant 0$ and $x \in \mathbf{R}$. □

Proposition 4.3. *For all predictable random fields Z, all $\beta > \mathrm{c}^2/2$, and all $k \geqslant 2$,*

$$\mathcal{N}_{\beta,k,\vartheta_\mathrm{c}}(p \circledast Z\dot{W}) \leqslant \mathrm{const}\cdot\sqrt{\frac{k}{2\beta-\mathrm{c}^2}} \cdot \mathcal{N}_{\beta,k,\vartheta_\mathrm{c}}(Z), \qquad (4.2)$$

where $\vartheta_\mathrm{c}(x) := \exp(\mathrm{c}x)$.

Proof. Note that

$$\|p_t\|^2_{L^2_{\vartheta_\mathrm{c}}(\mathbf{R})} \leqslant \sup_{z\in\mathbf{R}} p_t(z)\cdot\int_{-\infty}^\infty p_t(x)\mathrm{e}^{\mathrm{c}x}\,\mathrm{d}x = \frac{\mathrm{const}}{\sqrt{t}}\cdot\mathrm{e}^{\mathrm{c}^2 t/2},$$

whence

$$\int_0^\infty \mathrm{e}^{-\beta t}\|p_t\|^2_{L^2_{\vartheta_\mathrm{c}}(\mathbf{R})}\,\mathrm{d}t \leqslant \mathrm{const}\cdot\int_0^\infty \frac{\exp\left\{(\mathrm{c}^2-2\beta)t/2\right\}}{\sqrt{t}}\,\mathrm{d}t \qquad (4.3)$$

Proposition 4.2 completes the proof. □

Lemma 4.4. *If $\beta > c^2/4$, then*

$$\mathcal{N}_{\beta,k,\vartheta_c}(p_\bullet * u_0) < \mathcal{N}_{\beta,k,\vartheta_c}(u_0).$$

Proof. Clearly,

$$\begin{aligned}
\sqrt{\vartheta_c(x)}(p_t * u_0)(x) &= \int_{-\infty}^{\infty} \sqrt{\vartheta_c(y-x)}p_t(y-x)\sqrt{\vartheta_c(y)}u_0(y)\,\mathrm{d}y \\
&\leqslant \sup_{y\in\mathbf{R}}\left[\sqrt{\vartheta_c(y)}u_0(y)\right]\cdot\int_{-\infty}^{\infty}\sqrt{\vartheta_c(z)}p_t(z)\,\mathrm{d}z \\
&= \sup_{y\in\mathbf{R}}\left[\sqrt{\vartheta_c(y)}u_0(y)\right]\cdot\exp(c^2t/8).
\end{aligned}$$

Now multiply both sides by $\exp(-\beta t/2)$ and optimize. □

Proposition 4.5. *Suppose there exists $c \in \mathbf{R}$ such that $\sup_{x\in\mathbf{R}}|\mathrm{e}^{cx/2}u_0(x)| < \infty$, and let $\beta \geqslant \Theta c^2$ for a sufficiently large $\Theta > 1$. Then for all $k \geqslant 2$ there exists a finite constant $A_{\beta,k}$ such that $\mathrm{E}(|u_t(x)|^k) \leqslant A_{\beta,k}\exp(\beta t - cx)$, uniformly for all $t \geqslant 0$ and $x \in \mathbf{R}$.*

Proof. We begin by studying Picard's approximation to the solution u. Namely, let $u_t^{(0)}(x) := u_0(x)$, and then define iteratively

$$u_t^{(n+1)}(x) := (p_t * u_0)(x) + \left(p \circledast \left(\sigma\circ u^{(n)}\right)\dot{W}\right)_t(x),$$

for $t > 0$, $x \in \mathbf{R}$, and $n \geqslant 0$. Clearly,

$$\left\|u_t^{(n+1)}(x)\right\|_k \leqslant |(p_t * u_0)(x)| + \left\|\left(p \circledast \left(\sigma\circ u^{(n)}\right)\dot{W}\right)_t(x)\right\|_k,$$

whence for all $\beta > c^2/2$,

$$\mathcal{N}_{\beta,k,\vartheta_c}\left(u^{(n+1)}\right) < \text{const}\cdot\mathcal{N}_{\beta,k,\vartheta_c}(u_0) + \frac{\text{const}}{\sqrt{2\beta - c^2}}\cdot\mathcal{N}_{\beta,k,\vartheta_c}\left(u^{(n)}\right);$$

see Proposition 4.3 and Lemma 4.4. If Θ is sufficiently large then the coefficients of $\mathcal{N}_{\beta,k,\vartheta_c}(u_0)$ and $\mathcal{N}_{\beta,k,\vartheta_c}(u^{(n)})$ are both at most 1/2, whence it follows that

$$\mathcal{N}_{\beta,k,\vartheta_c}(u^{(n+1)}) \leqslant \mathcal{N}_{\beta,k,\vartheta_c}(u_0) = \sup_{x\in\mathbf{R}}\left|\mathrm{e}^{cx/2}u_0(x)\right|,$$

uniformly for all n. Now let $n \to \infty$ and apply Fatou's lemma to conclude that $\mathcal{N}_{\beta,k,\vartheta_c}(u) < \infty$. This is another way to state the conclusion. □

Proof of the assertion that $\overline{\lambda}(k) < \infty$. Since u_0 undergoes exponential decay at infinity, there exists $c > 0$ such that $\sup_{x\in\mathbf{R}}|\mathrm{e}^{\pm cx/2}u_0(x)| < \infty$. Consequently, $\mathrm{E}(|u_t(x)|^k) \leqslant A_{\beta,k}\exp(\beta t - c|x|)$ for $\beta > \Theta c^2$. That is,

$$\limsup_{t\to\infty}\frac{1}{t}\sup_{|x|\geqslant\alpha t}\log\mathrm{E}\left(|u_t(x)|^k\right) \leqslant \beta - c\alpha < 0,$$

provided that $\alpha > \beta/c$. That is, $\overline{\lambda}(k) < \alpha < \infty$ for such an α. □

Proof of the assertion that $\underline{\lambda}(k) > 0$. Let $J_0 := \inf_{z\in\mathbf{R}} |\sigma(z)/z|$ and note that

$$\|u_t(x)\|_2^2 \geqslant |(P_t u_0)(x)|^2 + J_0^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ \ |p_{t-s}(y-x)|^2 \, \|u_s(y)\|_2^2. \tag{4.4}$$

Let us define for all $\alpha, \beta > 0$, the following norms for an arbitrary random field $v := \{v_t(x)\}_{t\geqslant 0, x\in\mathbf{R}}$:

$$\begin{aligned}
\mathcal{N}^+_{\alpha,\beta}(v) &:= \left[\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \int_{\substack{x\in\mathbf{R}:\\ x\geqslant \alpha t}} \mathrm{d}x \ \ \|v_t(x)\|_2^2\right]^{1/2}, \\
\mathcal{N}^-_{\alpha,\beta}(v) &:= \left[\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \int_{\substack{x\in\mathbf{R}:\\ x\leqslant -\alpha t}} \mathrm{d}x \ \ \|v_t(x)\|_2^2\right]^{1/2}, \\
\mathcal{N}_{\alpha,\beta}(v) &:= \left[\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \int_{|x|\geqslant \alpha t} \mathrm{d}x \ \ \|v_t(x)\|_2^2\right]^{1/2} \\
&= \left[\left(\mathcal{N}^+_{\alpha,\beta}(v)\right)^2 + \left(\mathcal{N}^-_{\alpha,\beta}(v)\right)^2\right]^{1/2}.
\end{aligned}$$

If $x, y \in \mathbf{R}$ and $0 \leqslant s \leqslant t$, then the triangle inequality implies that

$$\mathbf{1}_{[\alpha t,\infty)}(x) \geqslant \mathbf{1}_{[\alpha(t-s),\infty)}(x-y)\cdot \mathbf{1}_{[\alpha s,\infty)}(y). \tag{4.5}$$

For all $r \geqslant 0$, let

$$T_\alpha(r) := \int_{\substack{z\in\mathbf{R}:\\ z\geqslant \alpha r}} |p_r(z)|^2 \ \mathrm{d}z \overset{\text{(symmetry)}}{=} \int_{\substack{z\in\mathbf{R}:\\ z\leqslant -\alpha r}} |p_r(z)|^2 \ \mathrm{d}z,$$

and

$$S_\alpha(r) := \int_{\substack{y\in\mathbf{R}:\\ y\geqslant \alpha r}} \|u_r(y)\|_2^2 \,\mathrm{d}y.$$

According to (4.4) and (4.5),

$$\int_{x\geqslant \alpha t} \|u_t(x)\|_2^2 \,\mathrm{d}x \geqslant \int_{x\geqslant \alpha t} |(p_t * u_0)(x)|^2 \ \mathrm{d}x + J_0^2 \cdot (T_\alpha * S_\alpha)(t). \tag{4.6}$$

We multiply both sides of (4.6) by $\exp(-\beta t)$ and integrate $[\mathrm{d}t]$ to find

$$\begin{aligned}
\left|\mathcal{N}^+_{\alpha,\beta}(u)\right|^2 &\geqslant \left|\mathcal{N}^+_{\alpha,\beta}(p_\bullet * u_0)\right|^2 + J_0^2\cdot \widetilde{T}_\alpha(\beta)\widetilde{S}_\alpha(\beta) \\
&= \left|\mathcal{N}^+_{\alpha,\beta}(p_\bullet * u_0)\right|^2 + J_0^2\cdot \widetilde{T}_\alpha(\beta)\left|\mathcal{N}^+_{\alpha,\beta}(u)\right|^2,
\end{aligned} \tag{4.7}$$

where $\widetilde{H}(\beta) := \int_0^\infty \exp(-\beta t) H(t)\,\mathrm{d}t$ defines the Laplace transform of H for every measurable function $H : \mathbf{R}_+ \to \mathbf{R}_+$. Also, we can apply a similar argument, run on the negative half of the real line, to deduce that

$$\left|\mathcal{N}^-_{\alpha,\beta}(u)\right|^2 \geqslant \left|\mathcal{N}^-_{\alpha,\beta}(p_\bullet * u_0)\right|^2 + J_0^2\cdot \widetilde{T}_\alpha(\beta)\left|\mathcal{N}^-_{\alpha,\beta}(u)\right|^2. \tag{4.8}$$

Next we add the inequalities (4.7) and (4.8) to conclude that

$$\left|\mathcal{N}_{\alpha,\beta}(u)\right|^2 \geqslant \left|\mathcal{N}_{\alpha,\beta}(p_\bullet * u_0)\right|^2 + J_0^2 \cdot \widetilde{T}_\alpha(\beta) \left|\mathcal{N}_{\alpha,\beta}(u)\right|^2.$$

Next we may observe that $|\mathcal{N}_{\alpha,\beta}(p_\bullet * u_0)| > 0$. This holds because $u_0 \geqslant 0$, $u_0 > 0$ on a set of positive measure, and u_0 is lower semicontinuous. Indeed, if it were not so, then $\int_{|x|\geqslant \alpha t}(p_t * u_0)(x)\,\mathrm{d}x = 0$ for almost all, hence all, $t > 0$. But then we would let $t \to 0$ to deduce from this and FATOU's lemma that $\int_{-\infty}^{\infty} u_0(x)\,\mathrm{d}x = 0$, which is a contradiction.

The preceding development implies the following:

$$\text{If } \mathcal{N}_{\alpha,\beta}(u) < \infty, \text{ then } \widetilde{T}_\alpha(\beta) < J_0^{-2}. \tag{4.9}$$

By the monotone convergence theorem,

$$\lim_{\alpha\downarrow 0} \widetilde{T}_\alpha(\beta) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-\beta t}\|p_t\|_{L^2(\mathbf{R})}^2\,\mathrm{d}t \propto \beta^{-1/2} \qquad \text{for all } \beta > 0.$$

Let $\beta \downarrow 0$ to conclude that $\widetilde{T}_\alpha(\beta) > J_0^{-2}$ for all sufficiently-small positive α and β. In light of (4.9), this completes our demonstration. □

5 Lecture 5: Fixed-time results

In this lecture we sometimes add a parameter to our stochastic heat equation. Namely, we consider

$$\left|\begin{aligned}&\partial_t u_t(x) = \frac{\kappa}{2}\partial^2_{xx} u_t(x) + \sigma(u_t(x))\dot{W}_t(x),\\ &\text{subject to } u_0 \text{ being bounded, measurable, and nonrandom,}\end{aligned}\right. \tag{5.1}$$

where $\kappa > 0$ is a fixed constant. The proof of Theorem 3.1 can be easily adjusted to show that (5.1) has a unique solution provided that $\mathrm{Lip}_\sigma < \infty$.

It is natural to view the solution, dynamically, as a stochastic process u_t that takes values in a suitable space of random functions. "Intermittency" gives information about u_t for large values of t. Let us say a few things about the behavior of u_t for "typical" values of t. I will only state results—without proofs—as the proofs are somewhat long, and instead include pointers to the literature wherein you can find the details of the arguments.

5.1 Chaotic behavior. Our first fixed-time result is that when u_0 has compact support, u_t is a bounded function for all $t > 0$.

Theorem 5.1 (FOONDUN and K [21]). *If $\sigma(0) = 0$, $\inf_{z\in\mathbf{R}} |\sigma(z)/z| > 0$, and $u_0 : \mathbf{R} \to \mathbf{R}_+$ is Lipschitz continuous with compact support, then for all $t > 0$:*

$$\sup_{x\in\mathbf{R}} u_t(x) = \sup_{x\in\mathbf{R}} |u_t(x)| < \infty \quad a.s.$$

In fact, $\sup_{x\in\mathbf{R}} u_t(x) \in L^k(\Omega)$ for all $k \in [2\,,\infty)$.

Idea of the proof. Choose and fix a $t > 0$ throughout. Since $\sigma(0) = 0$ and $u_0 \geqslant 0$, MUELLER's comparison principle tells us that $u_t \geqslant 0$ a.s. In particular, $\sup_{x\in\mathbf{R}} |u_t(x)| = \sup_{x\in\mathbf{R}} u_t(x)$. Moreover, the mild formulation (3.2) of u shows that if u_0 is supported in $[-c\,,c]$, then [keeping in mind some $t > 0$ that is fixed],

$$\begin{aligned}\mathrm{E}|u_t(x)| &= \mathrm{E}u_t(x) = (p_t * u_0)(x)\\ &\leqslant \frac{\sup_{z\in\mathbf{R}}[u_0(z)]}{\sqrt{2\pi t}} \int_{-c}^{c} \exp\left(-\frac{|x-y|^2}{2t}\right) \mathrm{d}y\\ &\leqslant \text{const}\cdot\exp\left(-\frac{x^2}{4t}\right).\end{aligned} \tag{5.2}$$

since $|x-y|^2 \geqslant \frac12 x^2 - c^2$ whenever $|y| \leqslant c$.

Therefore, one might imagine that with probability one $\lim_{|x|\to\infty} u_t(x) = 0$. If so, then we would have desired result by continuity.

Let us argue why $\lim_{x\to\infty} u_t(x)$ has to be zero a.s. A similar argument will show that $\lim_{x\to-\infty} u_t(x) = 0$ also.

Let $x_k := (2tq\log k)^{1/2}$ for all integers $k \geqslant 1$, where $q > 1$ is fixed. According to (5.2), $\mathrm{E}|u_t(x_k)| = O(k^{-q})$, whence $\lim_{k\to\infty} u_t(x_k) = 0$ a.s., thanks to the Borel–Cantelli lemma. Since $x_{k+1} - x_k \sim \text{const}\cdot[k\sqrt{\log k}]^{-1}$ for k large, one might imagine that therefore $\sup_{x\in[x_k,x_{k+1}]} |u_t(x) - u_t(x_k)| \approx 0$ whenever $k \gg 1$. This is the case, and clearly implies the result. The previous assertion requires an "asymptotic modulus of continuity argument," which is basically a refinement of KOLMOGOROV's continuity theorem of the general theory of stochastic processes. □

By contrast with Theorem 5.1, if u_0 is bounded uniformly away from zero, then the solution is unbounded. Moreover, and this has some connections to statistical physics, the solution grows at a predescribed rate in that case. This sensitive dependence on the initial data is evidence of "chaotic behavior," a term which best remains rigorously undefined.

Theorem 5.2 (CONUS, JOSEPH, and K [12]). *If $\sigma(x) = x$ and $\inf_{x\in\mathbf{R}} u_0(x) > 0$, then for all $t > 0$ there exists a finite constant $c(t) > 1$, such that*

$$\frac{1}{c(t)\kappa^{\delta}} \leqslant \liminf_{R\to\infty} \sup_{|x|<R} \frac{\log u_t(x)}{(\log R)^{\sigma}} \leqslant \limsup_{R\to\infty} \sup_{|x|<R} \frac{\log u_t(x)}{(\log R)^{\sigma}} \leqslant \frac{c(t)}{\kappa^{\delta}},$$

where $\sigma := 2/3$ and $\delta := 1/3$.

Remark 5.3. We might notice the following consequence:

$$\frac{1}{c(t)\kappa^{\delta}} \leqslant \limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\sigma}} \leqslant \frac{c(t)}{\kappa^{\delta}} \qquad \text{almost surely for all } t>0.$$ □

Remark 5.4. Let u solve the stochastic heat equation (5.1) and $\sigma(x) := x$. Define $h_t(x) := \log u_t(x)$ [a "COLE–HOPF transformation"]; then $u_t(x) = \exp(h_t(x))$, and an informal computation shows that

$$\partial_t u_t(x) = e^{h_t(x)}\partial_t h_t(x), \;\; \partial_{xx} u_t(x) = e^{h_t(x)}\left(\partial_{xx} h_t(x) + (\partial_x h_t(x))^2\right).$$

In other words, $h_t(x)$ is the COLE–HOPF solution to the SPDE, which is described the following:[6]

$$\partial_t h_t(x) = \frac{\kappa}{2}\partial_{xx} h_t(x) + \frac{\kappa}{2}\left(\partial_x h_t(x)\right)^2 + \dot{W}_t(x).$$

This is the celebrated *KPZ equation* [29]—so named after KARDAR, PARISI, and ZHANG—and Theorem 5.2 asserts that

$$\frac{1}{c(t)\kappa^{\delta}} \leqslant \liminf_{R\to\infty} \sup_{|x|<R} \frac{h_t(x)}{(\log R)^{\sigma}} \leqslant \limsup_{R\to\infty} \sup_{|x|<R} \frac{h_t(x)}{(\log R)^{\sigma}} \leqslant \frac{c(t)}{\kappa^{\delta}},$$

for $\sigma := 2/3$ and $\delta := 1/3$. We may think of σ as a spatial scaling exponent and δ as a temporal [or diffusive] one. The relation "$2\sigma = 1 + \delta$," valid in this context, is the socalled "KPZ relation," after ref. [29], where it has been

[6]The KPZ equation is an informal equation, though great strides have been made recently to make rational sense of this equation in various related contexts [3, 26, 27].

predicted for various models of statistical mechanics [including a large-time-fixed-space version of this one]. This relation has recently been verified for a number of related models of statistical mechanics; see, in particular, ALBERTS ET AL [1] and CONUS ET AL [13]. □

Some ideas for the proof of Theorem 5.2. MUELLER's comparison principle reduces the problem to the case that u_0 is identically a constant, say $u_0 \equiv 1$. At this point there are two key main steps behind the proof of Theorem 5.2.

Step 1. First, that there exists $c = c_t \in (1\,,\infty)$ such that for all $\lambda > 1$,

$$c^{-1} \exp\left(-c|\log\lambda|^{3/2}\right) \leqslant \mathrm{P}\left\{u_t(x) \geqslant \lambda\right\} \leqslant c\exp\left(-c^{-1}|\log\lambda|^{3/2}\right). \tag{5.3}$$

[The distribution of $u_t(x)$ does not depend on x, since $u_0 \equiv 1$. A hint to how one would prove this: Use PICARD's iteration method, and show that every step of this approximation has the property that its law at the space-time point $(x\,,t)$ does not depend on x.]

One can prove the following variation of CHEBYSHEV's inequality:

Side Lemma: *If Z is a positive random variable for which there exists $A \in (1\,,\infty)$ such that $\mathrm{E}[Z^k] \leqslant A\exp(Ak^3)$ for all $k \geqslant 2$, then there exists $B \in (1\,,\infty)$ such that*

$$\mathrm{P}\{Z > \lambda\} \leqslant B\exp\left(-B^{-1}|\log\lambda|^{3/2}\right) \qquad \text{for all } \lambda > 1.$$

I will leave this for you to prove on your own. [This is a nice exercise.] From it we can conclude that our moment estimates for $u_t(x)$ yield the upper probability bound in (5.3).

The lower bound uses an old idea of PALEY and ZYGMUND [38]. Namely, that *if $Z > 0$ then*

$$4\mathrm{P}\left\{Z \geqslant 2^{-1/k}\|Z\|_k\right\} \geqslant \|Z\|_k^{2k}\|Z\|_{2k}^{-2k}.$$

Here is the quick proof: For every $\mu > 0$,

$$\mathrm{E}(Z^k) \leqslant \mu^k + \mathrm{E}(Z^k;\, Z > \mu) \leqslant \mu^k + \sqrt{\mathrm{E}(Z^{2k}) \cdot \mathrm{P}\{Z > \mu\}}.$$

Solve this with $\mu := [(1/2)\mathrm{E}(Z^k)]^{1/k}$ to finish.

We may apply the Paley–Zygmund inequality to $Z := |u_t(x)|^k$, using the following bounds, rigorously derived first by BERTINI and CANCRINI [4]: (i) $\mathrm{E}(|u_t(x)|^k) \geqslant c\exp(ck^3t)$ for $c \in (0\,,1)$; and (ii) $\mathrm{E}(|u_t(x)|^{2k}) \leqslant C\exp(Ck^3t)$ for $C > 1$. In this way we find that

$$\begin{aligned} 4\mathrm{P}\left\{u_t(x) \geqslant (c/2)^{1/k}\mathrm{e}^{ck^2t}\right\} &\geqslant 4\mathrm{P}\left\{u_t(x) \geqslant 2^{-1/k}\|u_t(x)\|_k\right\} \\ &\geqslant \|u_t(x)\|_k^{2k}\|u_t(x)\|_{2k}^{-2k} \geqslant (c^2/C)\exp(-Ck^3t). \end{aligned}$$

Set $\lambda := 2^{-k}c^{1/k}\exp(ck^2t)$ and solve the preceding inequality for λ large [so that $k \leqslant \text{const}\cdot|\log\lambda|^{1/2}$, whence $\exp(Ck^3t) \leqslant \exp(\text{const}\cdot|\log\lambda|^{3/2})$] to obtain the probability lower bound.

Step 2. We would like to show that if x and x' are "sufficiently far apart," then $u_t(x)$ and $u_t(x')$ are "sufficiently independent." Once this is done, the result of Step 1 and the Borel–Cantelli lemma together do the job.

Our goal is achieved by coupling: We use the same white noise $\dot{W}$, and consider the solution to the random integral equation:

$$u_t^{(\beta)}(x) := 1 + \int_{\substack{(s,y)\in(0,t)\times\mathbf{R}:\\ |y-x|\leqslant\beta}} p_{t-s}(y-x)\sigma\left(u_s^{(\beta)}(y)\right)\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y.$$

One proves the existence and uniqueness of the solution $u^{(\beta)}$ as one would for an SPDE [though, strictly speaking, the preceding is not an SPDE]. Moreover, one can show that if β is sufficiently large, then $u^{(\beta)} \approx u$. Choose and fix $\beta \gg 1$ so large that it ensures that $u^{(\beta)}$ is "sufficiently close" to u. Once done carefully, our argument will reduce our problem to the following claim: If x and x' are "sufficiently close," then $u_t^{(\beta)}(x)$ and $u_t^{(\beta)}(x')$ are "almost independent."

In order to accomplish this we proceed with a second coupling argument: Define $u^{(\beta,n)}$ to be the nth step of Picard's iteration approximation to $u^{(\beta)}$. That is, $u_t^{(\beta,0)}(x) := 1$, and

$$u_t^{(\beta,n+1)}(x) := 1 + \int_{\substack{(s,y)\in(0,t)\times\mathbf{R}:\\ |y-x|\leqslant\beta}} p_{t-s}(y-x)\sigma\left(u_s^{(\beta,n)}(y)\right)\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y.$$

As part of the existence/uniqueness proof of $u^{(\beta)}$, we show that $u^{(\beta,n)} \approx u^{(\beta)}$ if n is "sufficiently large." Now you should convince yourself that if $|x-x'| > 2n\beta$ then $u_t^{(\beta,n)}(x)$ and $u_t^{(\beta,n)}(x')$ are [*exactly*] independent. To finish this argument it remains to make precise what "sufficiently large" means throughout. This can be done by performing very careful [though somewhat tedious] moment computations. □

5.2 Fractal-like exceedance sets. Suppose $u_t(x)$ solves (3.1) once again, and define exeedance sets,

$$E_\alpha(R) := \left\{x \in [0\,,R]:\ u_t(x) \geqslant \exp\left(\alpha(\log R)^{2/3}\right)\right\},$$

where $t > 0$ is fixed here and throughout. As it turns out, $x \mapsto u_t(x)$ is a.s. continuous (Walsh [43, Ch. 3]). Therefore, every $E_\alpha(R)$ is a random closed subset of $[0\,,R]$ for every $R > 0$.

Theorem 5.2 implies that: (i) If α is too small then $E_\alpha(R)$ is eventually unbounded a.s. as $R \to \infty$; and (ii) If α is too large, then $E_\alpha(R)$ is eventually empty as $R \to \infty$ a.s.

Open Problem. Scale $E_\alpha(R)$ so that it is a set in $[0\,,1]$; i.e.,

$$F_\alpha(R) := \left\{x \in [0\,,1]:\ u_t(Rx) \geqslant \exp\left(\alpha(\log R)^{2/3}\right)\right\}.$$

Is it true that $F_\alpha(R)$ a.s. "converges" to a [random] set $F_\alpha \subset [0\,,1]$ as $R \to \infty$?

One might imagine that if $F_\alpha(R)$ did converge to some random set F_α, then it would do so nicely, and F_α would have to be a random fractal. Moreover, if $\dim_{_H} F_\alpha$ denotes a "fractal dimension" for that fractal F_α, then one might also expect that $\lim_{R\to\infty} \log|F_\alpha(R)|/\log R = -\dim_{_H} F_\alpha$ a.s., where $|\cdots|$ denotes Lebesgue measure. In light of this discussion, the following suggests that $F(\alpha)$ is likely to be fractal like with non-trivial "fractal dimension."

Theorem 5.5 (Conus, Joseph, and K [11]). *There exists $\alpha_* > 0$ such that for all $\alpha \in (0\,,\alpha_*)$,*

$$-1 < \liminf_{R\to\infty} \frac{\log|F_\alpha(R)|}{\log R} \leqslant \limsup_{R\to\infty} \frac{\log|F_\alpha(R)|}{\log R} < 0 \qquad \textit{a.s.}$$

Open Problem. Does $\delta := \lim_{R\to\infty} \log|F_\alpha(R)|/\log R$ exist?

5.3 Intermittency islands. Consider the following non-linear stochastic heat equation, where $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous:

$$\left|\begin{array}{l} \partial_t u_t(x) = \frac12 \partial^2_{xx} u_t(x) + \sigma(u_t(x))\dot W_t(x), \\ \text{subject to } u_0(x) \equiv 1 \text{ for all } x \in \mathbf{R}. \end{array}\right.$$

Since $\int_{-\infty}^\infty p_t(y-x)u_0(y)\,\mathrm{d}y = 1$, the solution u can be written, in mild form, as follows:

$$u_t(x) = 1 + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\dot W_s(y)\,\mathrm{d}s\,\mathrm{d}y. \tag{5.4}$$

We have seen that if σ grows roughly linearly, then the solution tends to develop tall peaks. I conclude these lectures by presenting an estimate for the number of peaks. With this aim in mind, let us choose and fix a time $t > 0$.

Definition 5.6. Choose and fix two numbers $0 < a < b$, and a time $t > 0$. We say that a closed interval $I \subset \mathbf{R}_+$ is an $(a\,,b)$-*island* if: (i) $u_t(\inf I) = u_t(\sup I) = a$; (ii) $u_t(x) > a$ for all $x \in I^\circ$; and (iii) $\sup_{x\in I} u_t(x) > b$. Let $J_t(a\,,b\,;R)$ denote the length of the largest $(a\,,b)$-island inside the interval $(0\,,R)$.

Theorem 5.7 (Conus, Joseph, and K [11]). *If $\sigma(1) \neq 0$, $1 < a < b$ and $\mathrm{P}\{u_t(0) > b\} > 0$, then*

$$\limsup_{R\to\infty} \frac{J_t(a\,,b\,;R)}{(\log R)^2} < \infty \qquad \textit{a.s.}$$

If, in addition, σ is bounded then

$$\limsup_{R\to\infty} \frac{J_t(a\,,b\,;R)}{\log R\cdot(\log\log R)^{3/2}} < \infty \qquad a.s.$$

The idea is that to estimate the correlation length of $x \mapsto u_t(x)$ very carefully, and then use coupling to relate the islands to the longest-run problem in coin tossing (Erdős and Rényi [20]). The proof is somewhat technical. Therefore, instead of hashing that out, let me conclude by making a few related remarks:

1. The condition that $\sigma(1) \neq 0$ is necessary. Indeed, if $\sigma(1) = 0$ then $u_t = 1$ [because $u_0 \equiv 1$]. In other words, if $\sigma(1) = 0$ then there are initial functions for which the solution to the heat equation is bounded. In those cases any discussion of tall islands is manifestly moot.

2. In order to see the condition that $\mathrm{P}\{u_t(0) > b\} > 0$ is non vacuous, we suppose to the contrary that $\mathrm{P}\{u_t(0) > b\} = 0$ for all $b > 1$ and derive a contradiction as follows: Since $\mathrm{E}u_t(0) = 1$, there must exist $b \geqslant 1$ such that $\mathrm{P}\{u_t(0) \geqslant b\} > 0$. Therefore, it must be that $\mathrm{P}\{u_t(0) > 1\} = 0$, whence $u_t(0) = 1$ a.s. This and (5.4) together show that

   $$\int_{(0,t)\times\mathbf{R}} p_{t-s}(y)\sigma(u_s(y))\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y = 0 \qquad \text{a.s.}$$

   But $M_\tau := \int_0^\tau p_{t-s}(y)\sigma(u_s(y))\dot{W}_s(y)\,\mathrm{d}s\,\mathrm{d}y$ $(0 \leqslant \tau \leqslant t)$ defines a mean-zero continuous L^2 martingale. Therefore, its quadratic variation must be zero. In particular,

   $$\int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\ [p_{t-s}(y)\sigma(u_s(y))]^2 = 0 \qquad \text{a.s.}$$

   The heat kernel never vanishes; therefore, $\sigma(u_s(y)) = 0$ a.s. for almost all $s \in (0\,,t)$ and $y \in \mathbf{R}$, whence $\sigma(u_s(y)) = 0$ a.s. for all $s \in (0\,,t)$ and $y \in \mathbf{R}$, by continuity. Let $s \downarrow 0$ to deduce that $\sigma(u_0(y)) = 0$, whence $u_0(y) \neq 1$. This is a contradiction.

Open Problem. Are there any nontrivial lower bounds on the lim sup of $J_t(a\,,b\,;R)$? For instance, is it true that $\limsup_{R\to\infty} J_t(a\,,b\,;R) > 0$ a.s.?

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