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Probability Theory and Related Fields

Stochastic p.d.e.'s arising from the long range contact and long range voter processes

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Summary. A long range contact process and a long range voter process are scaled so that the distance between sites decreases and the number of neighbors of each site increases. The approximate densities of occupied sites, under suitable time scaling, converge to continuous space time densities which solve stochastic p.d.e.'s. For the contact process the limiting equation is the Kolmogorov–Petrovskii–Piscuinov equation driven by branching white noise. For the voter process the limiting equation is the heat equation driven by Fisher–Wright white noise.

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1 Introduction

We define a sequence of one dimensional contact processes indexed by a parameter n. In the nth model the sites are indexed by $x \in n^{-2}\mathbb{Z}$. We label the state of site x by $\xi_t^n(x)$ where $\xi_t^n(x) = 1$ if the site is occupied at time t and $\xi_t^n(x) = 0$ if it is vacant. Two sites are neighbors, denoted by $x \sim y$, if $|x - y| \leqq n^{-1/2}$. Thus each site has $2c_1 n^{3/2}$ neighbors where $c_1(n) \to 1$ as $n \to \infty$. Occupied sites become vacant at rate n. Occupied sites also give birth at rate $n + \theta_c$. At the time of a birth at site x, a site is chosen uniformly from the neighbors of x and, if vacant, becomes occupied. The parameter $\theta_c \in \mathbb{R}$ (where the subscript stands for contact process) is fixed throughout and we consider only $n \geqq |2\theta_c|$.

An approximate density is defined by $A_c(\xi_t^n)$ where for f: $n^{-2}\mathbb{Z} \to \mathbb{R}$

$$A_c(f)(x) := (2c_1 n^{1/2})^{-1} \sum_{y \sim x} f(y) \quad \text{for } x \in n^{-2}\mathbb{Z}\,.$$

We linearly interpolate between sites to obtain a function $A_c(\xi_t^n)(x)$ for $x \in \mathbb{R}$. Let $e_\lambda(x) = \exp(\lambda|x|)$ for $\lambda \in \mathbb{R}$. The set

$$\mathscr{C} = \{f\colon \mathbb{R} \to [0, \infty) \text{ continuous with } |f(x)e_\lambda(x)| \to 0 \text{ as } |x| \to \infty \text{ for all } \lambda < 0\}$$

is the set of non-negative continuous functions with slower than exponential growth. Define $\|f\|_\lambda = \sup_x |f(x)e_\lambda(x)|$ and give $\mathscr{C}$ the topology generated by the norms $(\|\ \ \|_\lambda\colon \lambda < 0)$. Then the paths $t \to A_c(\xi_t^n)$ are $\mathscr{C}$ valued (since, for instance, $|A_c(\xi_t^n)(x)| \leqq n$) and cadlag. We consider the law of $A_c(\xi^n)$ on the space of cadlag $\mathscr{C}$ valued paths with the Skorokhod topology.

Theorem 1 *Suppose that as $n \to \infty$ the initial conditions $(A_c(\xi_0^n)(x)\colon x \in \mathbb{R})$ converge (in $\mathscr{C}$) to $f_0 \in \mathscr{C}$. Then the approximate densities $(A_c(\xi_t^n)\colon t \geqq 0)$ converge in distribution as $n \to \infty$ to a continuous $\mathscr{C}$ valued process $(u_t\colon t \geqq 0)$ which solves the stochastic p.d.e. driven by white noise*

$$\partial_t u = (1/6)\Delta u + \theta_c u - u^2 + |2u|^{1/2}\dot{W} \tag{1.1}$$

with initial condition $u_0 = f_0$.

For the voter processes the lattice scale is different. In the nth model the voters are indexed by $x \in n^{-1}\mathbb{Z}$. The voters can take two opinions, labeled 0 and 1. Two voters are neighbors, again denoted by $x \sim y$, if $|x - y| \leqq n^{-1/2}$. Thus each voter has $2c_2 n^{1/2}$ neighbors where $c_2(n) \to 1$ as $n \to \infty$. Voters change their opinion at rate $O(n)$ and adopt the opinion of one of their neighbors. We allow a slight asymmetry where the opinion 1 is more dominant. More precisely, for each of the $O(n^{1/2})$ neighbors independently, they adopt the opinion of that neighbor at rate $n^{1/2}$ if it is 0 and at rate $n^{1/2} + \theta_v n^{-1/2}$ if it is 1. The parameter $\theta_v \geqq 0$ (where the subscript stands for voter process) is fixed througout and we consider only $n \geqq 2\theta_v$.

The opinion of the voter at x at time t is again labelled by $\xi_t^n(x) \in \{0, 1\}$. An approximate density is defined by $A_v(\xi_t^n)$ where for $f\colon n^{-1}\mathbb{Z} \to \mathbb{R}$

$$A_v(f)(x) := (2c_2 n^{1/2})^{-1} \sum_{y \sim x} f(y) \quad \text{for } x \in n^{-1}\mathbb{Z}\,.$$

We linearly interpolate between sites to obtain a function $A_v(\xi_t^n)(x)$ for $x \in \mathbb{R}$. Again the paths $t \to A_v(\xi_t^n)$ are cadlag $\mathscr{C}$ valued.

Theorem 2 *Suppose that as $n \to \infty$, the initial conditions, $(A_v(\xi_0^n)(x)\colon x \in \mathbb{R})$ converge (in $\mathscr{C}$) to $f_0 \in \mathscr{C}$. Then the approximate densities $(A_v(\xi_t^n)\colon t \geqq 0)$ converge in distribution as $n \to \infty$ to a continuous $\mathscr{C}$ valued process $(u_t\colon t \geqq 0)$ which solves the stochastic p.d.e. driven by white noise*

$$\partial_t u = (1/6)\Delta u + 2\theta_v u(1-u) + |4u(1-u)|^{1/2}\dot{W} \tag{1.2}$$

with initial condition $u_0 = f_0$.

Equations (1.1) and (1.2), where W is a space time white noise, can be given rigorous meaning in terms of an integral equation, as explained in Walsh [16, Chap. 3]. Uniqueness in law holds for both equations. Equation (1.1) without the overcrowding term $-u^2$ is the density of super Brownian motion with mass creation at rate θ_c. Uniqueness for this process is well known [3]. Uniqueness for (1.1) then follows from change of measure arguments (see [7] for the case where u_0 is integrable and [14] for the case $u_0 \in \mathscr{C}$). For (1.2) uniqueness follows from the existence of a dual process [12]. Solutions to both equations have been studied [8, 9, 14, 15]. They show phenomena (phase transition and travelling wave solutions) that are analogous to their underlying simple (not long range) discrete versions.

Theorem 1 was conjectured by R. Durrett and partially proved by Perkins [10] who showed that a discrete time process analogous to the long range contact process also converges to the limit (1.1). The method of proof here is similar. The models are shown to satisfy a martingale problem that approximates the martingale problem for the limiting processes. Tightness of the approximate densities is established. Passing to the limit in the approximate martingale problems, all limit points are shown to satisfy the limiting equations. The proof is then concluded by the uniqueness for solutions to the limiting equations. A stronger result is obtained here than in [10] which proved only convergence in distribution as finite measure valued processes.

Tightness is proved by estimating moments of small increments for the approximate densities and arguing as in the Kolmogorov tightness criterion. For this an approximate Green's function representation is established (equation (2.11)) for the approximate densities $A_c(\xi_t^n)(x)$ which is analogous to the Green's function representation for solutions to (1.1) but with certain error terms. Such a representation was also used in [10]. The method for estimating moments from the Green's function representation is analogous to that used for the limiting stochastic p.d.e.'s [13]. The extra work involved is to control the error terms. The same method works for the voter model and is considerably easier because the densities are known to be bounded by 1.

Finally we note that these random limits are possible because of the presence of an asymptotically critical branching mechanism. In one dimension, this allows a suitable rescaling to a stochastic p.d.e. driven by white noise. A well known example of this kind of scaling is the super Brownian motion, which is the limit of critical branching Brownian motions. In one dimension, the super Brownian motion has a density which satisfies a stochastic p.d.e. similar to those above. Super Brownian motion exists in higher dimensions. However it exists as a singular measure valued process and possible higher dimensional analogues for equations (1.1) and (1.2) are unclear.

The choice of uniform jumping mechanisms, although natural, is very special and the result should carry over to a class of jumping mechanisms. For instance symmetric jumps with the same variance and enough moments would allow the approximation by the local limit theorem that is needed.

Notation. C will denote a non-negative quantity whose dependence will be indicated but whose exact value is unimportant and may change from line to line. A point mass at $x \in \mathbb{R}$ is denoted δ_x. We write $p(t, x)$ for the Brownian transition density and P_t for the Brownian semigroup.

2 Long range contact process

For $f, g\colon n^{-2}\mathbb{Z} \to \mathbb{R}$ we write (f, g) for $n^{-2}\sum_x f(x)g(x)$ (whenever this sum is meaningful). If also v is a measure on $n^{-2}\mathbb{Z}$ we write (v, f) for $\int f\,dv$. We write again (with a slight abuse of notation) $\|f\|_\lambda$ for $\sup\{|f(x)e_\lambda(x)|: x \in n^{-2}\mathbb{Z}\}$. We also use the notation, for $x \in n^{-2}\mathbb{Z}$, $\delta > 0$

$$D(f, \delta)(x) = \sup\{|f(y) - f(x)|: |y - x| \leqq \delta,\ y \in n^{-2}\mathbb{Z}\},$$

$$\Delta_c(f)(x) = (n + \theta_c)(2c_1 n^{3/2})^{-1} \sum_{y \sim x} (f(y) - f(x)).$$

The long range contact processes may be constructed by the graphical construction [4] from two independent families of i.i.d. Poisson processes:

$$(P_t(x)\colon x \in n^{-2}\mathbb{Z}) \quad \text{with rate } n,$$

$$(P_t(x, y)\colon x, y \in n^{-2}\mathbb{Z}, x \sim y) \quad \text{with rate } (2c_1 n^{3/2})^{-1}(n + \theta_c),$$

where the processes are indexed over $t \in [0, \infty)$. At a jump time of $P_t(x)$ the site x, if occupied, becomes vacant. At a jump time fo $P_t(x, y)$, if the site x is occupied, there is a birth and the site y, if vacant, becomes occupied. The dynamics of the contact process $\xi_t^n \in \{0, 1\}$ are captured in the equations

$$\xi_t^n(x) = \xi_0^n(x) - \int_0^t \xi_{s-}^n(x)\,dP_s(x) + \sum_{y \sim x} \int_0^t (1 - \xi_{s-}^n(x))\xi_{s-}^n(y)\,dP_s(y, x).$$

We define the measure valued process

$$v_t^n := n^{-1} \sum_x \delta_x I(\xi_t^n(x) = 1).$$

For most of the proof we now drop the superscripts and write simply ξ_t, v_t.

Step 1 An approximate martingale problem
Take a test function $\phi\colon [0, \infty) \times n^{-2}\mathbb{Z} \to \mathbb{R}$ with $t \to \phi_t(x)$ continuously differentiable and satisfying

$$\int_0^T (|\phi_s| + \phi_s^2 + |\partial_s \phi_s|, 1)\,ds < \infty. \tag{2.1}$$

Then, applying integration by parts to $\xi_t(x)\phi_t(x)$ and summing over x, for $t \leqq T$

$$
\begin{aligned}
(v_t, \phi_t) &= (v_0, \phi_0) + \int_0^t (v_s, \partial_s \phi_s)\, ds \\
&\quad - n^{-1} \sum_x \int_0^t \xi_{s-}(x) \phi_s(x)\, dP_s(x) \\
&\quad + n^{-1} \sum_x \sum_{y \sim x} \int_0^t (1 - \xi_{s-}(x)) \xi_{s-}(y) \phi_s(x)\, dP_s(y, x) \\
&= (v_0, \phi_0) + \int_0^t (v_s, \partial_s \phi_s)\, ds
\end{aligned}
$$

$$
+ n^{-1} \sum_x \int_0^t \xi_{s-}(x) \phi_s(x) \left(\left(\sum_{y \sim x} dP_s(x, y) \right) - dP_s(x) \right) \tag{2.2}
$$

$$
+ n^{-1} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(y) (\phi_s(x) - \phi_s(y))\, dP_s(y, x) \tag{2.3}
$$

$$
- n^{-1} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(x) \xi_{s-}(y) \phi_s(x)\, dP_s(y, x) . \tag{2.4}
$$

We break term (2.2) into two parts, an average term and a fluctuation term:

$$
\begin{aligned}
& n^{-1} \sum_x \int_0^t \xi_{s-}(x) \phi_s(x) \\
& \qquad \left(\left(\sum_{y \sim x} (dP_s(x, y) - (2c_1 n^{3/2})^{-1}(n + \theta_c)\, ds) \right) - (dP_s(x) - n\, ds) \right) \\
& \qquad + n^{-1} \theta_c \sum_x \int_0^t \xi_s(x) \phi_s(x)\, ds \\
& = Z_t(\phi) + \theta_c \int_0^t (v_s, \phi_s)\, ds
\end{aligned}
$$

where $Z_t(\phi)$ is the martingale defined by

$$
n^{-1} \sum_x \int_0^t \xi_{s-}(x) \phi_s(x) \left(\left(\sum_{y \sim x} (dP_s(x, y) - d\langle P(x, y) \rangle_s) \right) - (dP_s(x) - d\langle P(x) \rangle_s) \right)
$$

and has predictable brackets process given by

$$
\begin{aligned}
\langle Z(\phi) \rangle_t &= n^{-2} \sum_x \int_0^t \xi_{s-}(x) \phi_s^2(x) (2n + \theta_c)\, ds \\
&= (2 + \theta_c n^{-1}) \int_0^t (v_s, \phi_s^2)\, ds .
\end{aligned} \tag{2.5}
$$

We break term (2.3) into two parts, an average term and a fluctuation term:

$$n^{-1}\sum_x \sum_{y\sim x} \int_0^t \xi_{s-}(y)(\phi_s(x)-\phi_s(y))(dP_s(y,x)-d\langle P(y,x)\rangle_s)$$

$$+n^{-1}\sum_x \sum_{y\sim x} \int_0^t \xi_s(y)(\phi_s(x)-\phi_s(y))(2c_1 n^{3/2})^{-1}(n+\theta_c)\,ds$$

$$=E_t^{(1)}(\phi)+\int_0^t (v_s, \Delta_c\phi_s)\,ds$$

where the error term

$$E_t^{(1)}(\phi):=n^{-1}\sum_x \sum_{y\sim x} \int_0^t \xi_{s-}(y)(\phi_s(x)-\phi_s(y))(dP_s(y,x)-d\langle P(y,x)\rangle_s)$$

is a martingale with predictable brackets process given by

$$\text{(2.6)}\quad d\langle E^{(1)}(\phi)\rangle_t = n^{-2}\sum_x \sum_{y\sim x} \xi_t(y)(\phi_t(x)-\phi_t(y))^2(2c_1 n^{3/2})^{-1}(n+\theta_c)\,dt$$

$$\leqq (1+\theta_c n^{-1})(v_t, D^2(\phi_t, n^{-1/2}))\,dt$$

$$\leqq 2(v_t, e_{-2\lambda})\|D(\phi_t, n^{-1/2})\|_\lambda^2 dt \quad (\text{for any } \lambda)\,.$$

Alternatively we may bound

$$\text{(2.7)}\quad d\langle E^{(1)}(\phi)\rangle_t$$

$$\leqq n^{-2}\left(2c_1 n^{3/2}\right)^{-1}(n+\theta_c)\sum_x \sum_{y\sim x} \xi_t(y)\|\phi_t\|_0(|\phi_t(x)|+|\phi_t(y)|)\,dt$$

$$=\|\phi_t\|_0(1+\theta_c n^{-1})\left(n^{-1}\sum_y \xi_t(y)|\phi_t(y)|+n^{-2}\sum_x A_c(\xi_t)(x)|\phi_t(x)|\right)dt$$

$$\leqq 2\|\phi_t\|_0((v_t,|\phi_t|)+(A_c(\xi_t),|\phi_t|))\,dt\,.$$

We break term (2.4) into two parts, an average term and a fluctuation term:

$$n^{-1}\sum_x \sum_{y\sim x} \int_0^t \xi_{s-}(x)\xi_{s-}(y)\phi_s(x)(dP_s(y,x)-d\langle P(y,x)\rangle_s)$$

$$+n^{-1}\sum_x \sum_{y\sim x} \int_0^t \xi_s(x)\xi_s(y)\phi_s(x)(2c_1 n^{3/2})^{-1}(n+\theta_c)\,ds$$

$$=E_t^{(2)}(\phi)+(1+\theta_c n^{-1})\int_0^t (v_s, A_c(\xi_s)\phi_s)ds$$

where the error term

$$E_t^{(2)}(\phi):=n^{-1}\sum_x \sum_{y\sim x} \int_0^t \xi_{s-}(x)\xi_{s-}(y)\phi_s(x)(dP_s(y,x)-d\langle P(y,x)\rangle_s)$$

is a martingale with predictable brackets process given by

$$(2.8)\quad \langle E^{(2)}(\phi)\rangle_t = n^{-2}\sum_x\sum_{y\sim x}\int_0^t \xi_s(x)\xi_s(y)\phi_s^2(x)\left(2c_1n^{3/2}\right)^{-1}(n+\theta_c)ds$$

$$\leqq c_1^{-1}n^{-5/2}\sum_x\sum_{y\sim x}\int_0^t \xi_s(x)\xi_s(y)e_{-2\lambda}(x)\|\phi_s\|_\lambda^2 ds$$

$$\leqq c_1^{-1}n^{-5/2}e^{\lambda}\sum_x\sum_y\int_0^t \xi_s(x)\xi_s(y)e_{-\lambda}(x)e_{-\lambda}(y)\|\phi_s\|_\lambda^2 ds$$

$$= c_1^{-1}e^{\lambda}n^{-1/2}\int_0^t (v_s,e_{-\lambda})^2\|\phi_s\|_\lambda^2 ds \quad \text{(for any } \lambda).$$

Collecting terms we have the following semimartingale decomposition

$$(2.9)\quad (v_t,\phi_t) = (v_0,\phi_0) + \int_0^t (v_s, \partial_s\phi_s + \theta_c\phi_s + \Delta_c(\phi_s))ds$$

$$- \left(1+\theta_c n^{-1}\right)\int_0^t (v_s, A_c(\xi_s)\phi_s)ds + Z_t(\phi) + E_t^{(1)}(\phi) - E_t^{(2)}(\phi).$$

Step 2 Green's function representation
We now take a special test function in the above decomposition. For each $z \in n^{-2}\mathbb{Z}$ define a test function $\psi_t^z(x) \geqq 0$ (for $t \geqq 0$, $x \in n^{-2}\mathbb{Z}$) as the unique solution, satisfying the hypothesis (2.1), to

$$(2.10)\qquad \partial_t\psi_t^z = \Delta_c\psi_t^z$$

$$\psi_0^z(x) = \left(n^{1/2}/2c_1\right)I(x\sim z).$$

Note that Δ_c is the generator of a simple random walk X_t jumping at rate $n+\theta_c$ with symmetric steps of variance $((1/3)+o(1))n^{-1}$. Define

$$\bar{\psi}_t^z(x) = n^2P(X_t = x|X_0 = z).$$

Then $\psi_t^z(x) = (\psi_0^z, \bar{\psi}_t^x)$. Also the local central limit theorem implies that (when linearly interpolated) the functions $\psi_t^z(x), \bar{\psi}_t^z(x)$ converge to $p(t/3, z-x)$. We collect together the information we need about the test functions $\psi, \bar{\psi}$ below.

Lemma 3 *There exists $n_0 < \infty$ such that for $n \geqq n_0$, $T \geqq 0$, $z \in n^{-2}\mathbb{Z}$, $\lambda \geqq 0$*

(a) $(\psi_t^z, 1) = (\bar{\psi}_t, 1) = 1$ *and* $\|\psi_t^z\|_0 \leqq Cn^{1/2}$, $\|\bar{\psi}_t^z\|_0 \leqq Cn^2$ *for all* $t \geqq 0$,

(b) $(e_\lambda, \psi_t^z + \bar{\psi}_t^z) \leqq C(\lambda,T)e_\lambda(z)$ *for all* $t \leqq T$,

(c) $\|\psi_t^z\|_\lambda \leqq C(\lambda,T)(n^{1/2}\wedge t^{-2/3})e_\lambda(z)$ *for all* $t \leqq T$,

(d) $(|\psi_t^z - \psi_s^z|, 1) \leqq 6n|t-s|$ *for all* $s,t \geqq 0$

and for $n \geqq n_0$, $n^{-3/4} \leqq s < t \leqq T$, $y,z \in n^{-2}Z$, $|y-z| \leqq 1$,

(e) $\|\psi_t^z - \psi_t^y\|_\lambda \leqq C(\lambda,T)e_\lambda(z)(|z-y|^{1/2}t^{-1} + n^{-1/2}t^{-3/4})$,

(f) $\|\psi_t^z - \psi_s^z\|_\lambda \leqq C(\lambda,T)e_\lambda(z)(|t-s|^{1/2}s^{-3/2} + n^{-1/2}s^{-3/4})$,

(g) $\|D(\psi_t^z, n^{-1/2})\|_\lambda \leqq C(\lambda, T) e_\lambda(z) n^{-1/4} t^{-1}$.

The proof of Lemma 3, which uses estimates from the local limit theorem, is delayed until Sect. 4.

We apply (2.9) with the test function $\phi_s = e^{\theta_c(t-s)} \psi_{t-s}^x$ for $s \leqq t$, (it is straightforward to check that hypothesis (2.1) is satisfied). This test function is chosen so that the first drift term vanishes and the initial condition is chosen so that $(v_t, \phi_t) = (v_t, \psi_0^x) = A_c(\xi_t)(x)$. Thus we obtain an approximate Green's function representation for $A_c(\xi_t)$, for a fixed value of t.

$$
\begin{aligned}
(2.11) \qquad A_c(\xi_t)(x) &= (v_0, \psi_t^x) - (1 + \theta_c n^{-1}) \int_0^t (v_s, A_c(\xi_s) \psi_{t-s}^x)\, ds \\
&\quad + Z_t(\psi_{t-\cdot}^x) + E_t^{(1)}(\psi_{t-\cdot}^x) - E_t^{(2)}(\psi_{t-\cdot}^x)\,.
\end{aligned}
$$

We now use this Green's function representation to obtain some moment estimates needed for the proof of tightness.

Lemma 4 *Suppose that the initial conditions satisfy $A_c(\xi_0) \to f_0$ in $\mathscr{C}$ as $n \to \infty$. Then for $T \geqq 0$, $p \geqq 2$, $\lambda > 0$*

(a) $E(\sup_{t \leqq T} (v_t, e_{-\lambda})^p) \leqq C(f_0, \lambda, \theta_c, p, T)$

(b) $E(|E_t^{(i)}(\psi_{t-\cdot}^z)|^p) \leqq C(f_0, \lambda, \theta_c, p, T) n^{-p/16} e_{\lambda p}(z)$ *for all* $t \leqq T$

(c) $\|E(A_c^p(\xi_t))\|_{-\lambda p} \leqq C(f_0, \lambda, \theta_c, p, T)$ *for all* $t \leqq T$.

We shall need a technical lemma.

Lemma 5 *For* $f : n^{-2}\mathbb{Z} \to [0, \infty)$ *with* $(f, f) < \infty$, $\lambda \in \mathbb{R}$

(a) $(v_t, \psi_t^z) = (A_c(\xi_t), \bar{\psi}_t^z)$,

(b) $|(v_t, f) - (A_c(\xi_t), f)| \leqq \|D(f, n^{-1/2})\|_\lambda (v_t, e_{-\lambda})$,

(c) $(v_t, A_c(\xi_t) f) \leqq C(A_c^2(\xi_t), f) + C\|D(f, n^{-1/2})\|_\lambda (A^2(\xi_t), e_{-\lambda})$.

Proof of Lemma 5 Part (a) is straightforward and the proof is omitted. For part (b) we have

$$
\begin{aligned}
(A_c(\xi_t), f) &= \left(2c_1 n^{1/2}\right)^{-1} n^{-2} \sum_x \sum_{y \sim x} f(x) \xi_t(y) \\
&= \left(2c_1 n^{1/2}\right)^{-1} n^{-2} \sum_x \sum_{y \sim x} (f(x) - f(y))\, \xi_t(y) + (v_t, f)
\end{aligned}
$$

and

$$
\begin{aligned}
&|(2c_1 n^{1/2})^{-1} n^{-2} \sum_x \sum_{y \sim x} (f(x) - f(y))\, \xi_t(y)| \\
&\leqq \left(2c_1 n^{1/2}\right)^{-1} n^{-2} \sum_x \sum_{y \sim x} D\left(f, n^{-1/2}\right)(y) \xi_t(y) \\
&= \left(v_t, D\left(f, n^{-1/2}\right)\right) \\
&\leqq \|D\left(f, n^{-1/2}\right)\|_\lambda (v_t, e_{-\lambda})\,.
\end{aligned}
$$

In the next argument C will denote a strictly positive constant.

$$
\begin{aligned}
&\left(A^2(\xi_t), f\right) \\
&= \left(4c_1^2\right)^{-1} n^{-3} \sum_x \sum_{y \sim x} \sum_{y' \sim x} f(x) \xi_t(y) \xi_t(y') \\
&\leqq \left(4c_1^2\right)^{-1} n^{-3} \sum_x \sum_{y \sim x} \sum_{y' \sim x} (f(y) - D(f, n^{-1/2})(x)) \xi_t(y) \xi_t(y') \\
&\geqq C n^{-3} \sum_y \sum_{y' \sim y} \sum_x f(y) \xi_t(y) \xi_t(y') I(x \sim y, x \sim y') - (A^2(\xi_t), D(f, n^{-1/2})) \\
&\geqq C n^{-3/2} \sum_y \sum_{y' \sim y} f(y) \xi_t(y) \xi_t(y') - (A^2(\xi_t), D(f, n^{-1/2})) \\
&\geqq C(v_t, A_c(\xi_t) f) - \|D(f, n^{-1/2})\|_\lambda (A^2(\xi_t), e_{-\lambda})\,.
\end{aligned}
$$

Rearranging gives the desired bound for part (c). □

Proof of Lemma 4 Substituting $\phi_t = e_{-\lambda}$ into (2.9) gives

$$
(2.12) \qquad (v_t, e_{-\lambda}) \leqq (v_0, e_{-\lambda}) + \int_0^t (v_s, \theta_c e_{-\lambda} + \Delta_c e_{-\lambda})\, ds + M_t
$$

where M_t is a martingale with brackets

$$
\begin{aligned}
\langle M \rangle_t &\leqq C\left(\langle Z(e_{-\lambda}) \rangle_t + \langle E^{(1)}(e_{-\lambda}) \rangle_t + \langle E^{(2)}(e_{-\lambda}) \rangle_t\right) \\
&\leqq C\left(\int_0^t (v_s, e_{-2\lambda})\, ds + \int_0^t (v_s, e_{-2\lambda}) \|D(e_{-\lambda}, n^{-1/2})\|_\lambda^2 ds \right. \\
&\qquad \left. + n^{-1/2} e^\lambda \int_0^t (v_s, e_{-\lambda})^2 \|e_{-\lambda}\|_\lambda^2 ds \right) \\
&\leqq C(\lambda) \int_0^t 1 + (v_s, e_{-\lambda})^2 ds\,.
\end{aligned}
$$

It is straightforward to check that $\Delta_c e_{-\lambda} \leqq C(\lambda) e_{-\lambda}$. Lemma 5 part (b) gives

$$
(v_0, e_{-\lambda}) \leqq (1 + \|D(e_{-\lambda}, n^{-1/2})\|_\lambda)(A_c(\xi_0), e_{-\lambda}) \leqq C(\lambda)(A_c(\xi_0), e_{-\lambda})
$$

which is bounded by $C(f_0, \lambda)$ uniformly in n by the assumption on the initial conditions. We now apply a Burkholder–Davis–Gundy inequality in the form

$$
E(\sup_{s \leqq t} |X_s|^p) \leqq C(p) E(\langle X \rangle_t^{p/2} + \sup_{s \leqq t} |X_s - X_{s-}|^p)
$$

for a cadlag martingale X with $X_0 = 0$ (which may be derived from its discrete time version [2, Theorem 21.1]). Note that the largest possible jump of the martingales $Z_t(e_{-\lambda})$, $E_t^{(1)}(e_{-\lambda})$, $E_s^{(2)}(e_{-\lambda})$ is bounded by n^{-1}. Then taking pth powers in equation (2.12) and taking expectations, we have for $t \leqq T$

$$\begin{aligned} &E(\sup_{s\leqq t}(v_s, e_{-\lambda})^p) \\ &\leqq C(f_0,\lambda,p) + C(\lambda,\theta_c,p)E\left(\left(\int_0^t (v_s,e_{-\lambda})ds\right)^p\right) + C(p)E(\sup_{t\leqq T}|M_t|^p) \\ &\leqq C(f_0,\lambda,\theta_c,p,T)\left(1+\int_0^t E((v_s,e_{-\lambda})^p)ds\right) \\ &\quad + C(p)E(\langle M\rangle_t^{p/2}) + C(p)n^{-p} \\ &\leqq C(f_0,\lambda,\theta_c,p,T)\left(1+\int_0^t E((v_s,e_{-\lambda})^p)ds\right). \end{aligned}$$

Gronwall's inequality completes the proof of the part (a).

The largest possible jump of the martingales $Z_s(\psi_{t-\cdot}^z)$, $E_s^{(i)}(\psi_{t-\cdot}^z)$ is bounded (almost surely) by $n^{-1}\sup_{s\geqq 0}\|\psi_s^z\|_\infty \leqq Cn^{-1/2}$ by Lemma 3(a). Choose $t \leqq T$. Then by Burkholder's inequality and (2.8)

$$\begin{aligned} (2.13)\quad &E(|E_t^{(2)}(\psi_{t-\cdot}^z)|^p) \\ &\leqq C(p)n^{-p/2}E\left(\left(\int_0^t (v_s,e_{-\lambda})^2\|\psi_{t-s}^z\|_\lambda^2 ds\right)^{p/2}\right) + C(p)n^{-p/2} \\ &\leqq C(p)n^{-p/2}\left(1+\left(\int_0^t\|\psi_{t-s}^z\|_\lambda^2 ds\right)^{(p/2)-1}\int_0^t E((v_s,e_{-\lambda})^p)\|\psi_{t-s}^z\|_{-\lambda}^2 ds\right) \\ &\leqq C(f_0,\lambda,\theta_c,p,T)n^{-p/2}\left(1+\left(\int_0^t\|\psi_{t-s}^z\|_\lambda^2 ds\right)^{p/2}\right) \quad \text{(by part a)} \\ &\leqq C(f_0,\lambda,\theta_c,p,T)n^{-p/4}e_{\lambda p}(z) \quad \text{(using Lemma 3(c))}. \end{aligned}$$

Similarly using (2.7), Lemma 5(a) and Lemma 3(a–c)

$$\begin{aligned} (2.14)\quad &E(|E_t^{(1)}(\psi_{t-\cdot}^z)|^p) \\ &\leqq C(p)E\left(\left(\int_0^t\|\psi_{t-s}^z\|_0((v_s,\psi_{t-s}^z)+(A_c(\xi_s),\psi_{t-s}^z))ds\right)^{p/2}\right) + C(p)n^{-p/2} \\ &\leqq C(p,T)E\left(\left(\int_0^t (t-s)^{-2/3}(A_c(\xi_s),\psi_{t-s}^z+\bar\psi_{t-s}^z)ds\right)^{p/2}\right) + C(p)n^{-p/2} \\ &\leqq C(p,T)\int_0^t (t-s)^{-2/3}(E(A_c^{p/2}(\xi_s)),\psi_{t-s}^z+\bar\psi_{t-s}^z)ds + C(p)n^{-p/2} \end{aligned}$$

$$\leqq C(p,T)\int_0^t (t-s)^{-2/3}\|E(A_c^{p/2}(\xi_s))\|_{-\lambda p}(e_{\lambda p},\psi_{t-s}^z+\bar{\psi}_{t-s}^z)ds + C(p)n^{-p/2}$$
$$\leqq C(\lambda,p,T)e_{\lambda_p}(z)\int_0^t (t-s)^{-2/3}\|1+E(A_c^p(\xi_s))\|_{-\lambda p}ds + C(p)n^{-p/2}\,.$$

We return after proving part (c) to improve the bound on $E^{(1)}$.

$$\begin{aligned}(2.15)\qquad (v_0,\psi_t^z)^p &= (A(\xi_0),\bar{\psi}_t^z)^p\\ &\leqq \|A(\xi_0)\|_{-\lambda}^p(e_\lambda,\bar{\psi}_t^z)^p\\ &\leqq C\,(f_0,\lambda,p,T)\,e_{\lambda_p}(z)\end{aligned}$$

using the assumptions on the initial conditions and Lemma 3(b). Using Burkholder's inequality again and (2.5), we have

$$\begin{aligned}(2.16)\quad & E(|Z(\psi_{t-\cdot}^z)|^p)\\ &\leqq C(p)E\left(\left(\int_0^t (v_s,(\psi_{t-s}^z)^2)ds\right)^{p/2}\right)+C(p)n^{-p/2}\\ &\leqq C(\lambda,p,T)e_{\lambda_p}(z)\int_0^t (t-s)^{-2/3}\|1+E(A_c^p(\xi_s))\|_{-\lambda_p}ds + C(p)n^{-p/2}\end{aligned}$$

arguing as in (2.14). To prove part (c) we shall take pth powers and expectations in the Green's function representation (2.11). Collecting together the bounds (2.13–2.16) we have

$$\begin{aligned}&\|E\,(A_c^p(\xi_t))\,\|_{-\lambda p}\\ &= \sup_z E\,(|A_c(\xi_t)(z)|^p)\,e_{-\lambda p}(z)\\ &\leqq C\,(f_0,\lambda,\theta_c,p,T)\left(1+\int_0^t (t-s)^{-2/3}\|E\,(A_c^p(\xi_s))\,\|_{-\lambda_p}ds\right)\,.\end{aligned}$$

Then a slight modification of the usual Gronwall argument [16, Lemma 3.3] completes the proof of part (c).

To improve the bound on $E^{(1)}$ and finish the proof of part (b) we apply the same Burkholder argument but with a mixture of the two bounds (2.6 and 2.7) for the brackets process.

$$\begin{aligned}(2.17)\quad & E\left(|E_t^{(1)}\,(\psi_{t-\cdot}^z)\,|^p\right)\\ &\leqq C(p)E\left(\left(\int_{(t-n^{-3/8})_+}^t \|\psi_{t-s}^z\|_0\left((v_s,\psi_{t-s}^z)+(A(\xi_s),\psi_{t-s}^z)\right)ds\right)^{p/2}\right)\\ &+C(p)E\left(\left(\int_0^{(t-n^{-3/8})_+}(v_s,e_{-2\lambda})\,\|D(\psi_{t-s}^z,n^{-1/2})\|_\lambda^2 ds\right)^{p/2}\right)\\ &+C(p)n^{-p/2}\,.\end{aligned}$$

We may bound the first expectation on the right-hand side of (2.17) as above by

$$C(T)E\left(\left(\int_{(t-n^{-3/8})_+}^{t}(t-s)^{-2/3}(A(\xi_s),\psi_{t-s}^z+\bar{\psi}_{t-s}^z)\,ds\right)^{p/2}\right)$$

$$\leqq C(T)\left(\int_{(t-n^{-3/8})_+}^{t}(t-s)^{-2/3}(E(A^{p/2}(\xi_s)),\psi_{t-s}^z+\bar{\psi}_{t-s}^z)\,ds\right)$$

$$\left(\int_{(t-n^{-3/8})_+}^{t}(t-s)^{-2/3}(1,\psi_{t-s}^z+\bar{\psi}_{t-s}^z)\,ds\right)^{(p/2)-1}$$

$$\leqq C(T)\left(\int_{(t-n^{-3/8})_+}^{t}(t-s)^{-2/3}\|E(A^{p/2}(\xi_s))\|_{-\lambda p}(e_{\lambda p},\psi_{t-s}^z+\bar{\psi}_{t-s}^z)\,ds\right)$$

$$\left(\int_{(t-n^{-3/8})_+}^{t}(t-s)^{-2/3}(1,\psi_{t-s}^z+\bar{\psi}_{t-s}^z)\,ds\right)^{(p/2)-1}$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)\,n^{-p/16}e_{\lambda p}(z)\quad\text{(using part c)}\,.$$

We may bound the second expectation on the right-hand side of (2.17) by

$$C(p)\left(\int_0^{(t-n^{-3/8})_+}E((v_s,e_{-2\lambda})^{p/2})\|D\left(\psi_{t-s}^z,n^{-1/2}\right)\|_\lambda^2 ds\right)$$

$$\left(\int_0^{(t-n^{-3/8})_+}\|D\left(\psi_{t-s}^z,n^{-1/2}\right)\|_\lambda^2 ds\right)^{(p/2)-1}$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)\left(\int_0^{(t-n^{-3/8})_+}\|D\left(\psi_{t-s}^z,n^{-1/2}\right)\|_\lambda^2 ds\right)^{p/2}\quad\text{(using part a)}$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)\,e_{\lambda p}(z)n^{-p/4}\left(\int_0^{(t-n^{-3/8})_+}(t-s)^{-2}ds\right)^{p/2}\quad\text{(Lemma 3(f))}$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)\,e_{\lambda p}(z)n^{-p/16}\,.$$

The two bounds together complete the proof of part (b). □

Step 3 Tightness

We now use the Green's function representation to estimate moment differences for the approximate densities. We assume throughout this section that $A(\xi_0)\to f_0$ in $\mathscr{C}$. Define

$$\hat{A}_c(\xi_t)(z)=A_c(\xi_t)(z)-(v_0,\psi_t^z)\,.$$

Note that this is natural given the Green's function representation.

Lemma 6 *For* $0\leqq s\leqq t\leqq T$, $y,z\in n^{-2}\mathbb{Z}$, $|t-s|\leqq 1$, $|y-z|\leqq 1$, $\lambda>0$, $p\geqq 2$

$$E(|\hat{A}_c(\xi_t)(z)-\hat{A}_c(\xi_s)(y)|^p)\leqq C(\lambda,p,\theta_c,T,f_0)e_{\lambda_p}(z)(|t-s|^{p/24}+|z-y|^{p/24}+n^{-p/12})\,.$$

Proof. The idea in the proof is to mimic the similar moment estimates for the solutions to equations like (1.1) (see for example [13]). Fix $s, t, T, y, z, \lambda, p$ as in the statement. We decompose the increment into a space increment $\hat{A}_c(\xi_t)(z) - \hat{A}_c(\xi_t)(y)$ and a time increment $\hat{A}_c(\xi_t)(y) - \hat{A}_c(\xi_s)(y)$. We shall subtract the Green's function representations for $\hat{A}_c(\xi_t)(z)$ and $\hat{A}_c(\xi_t)(y)$, take pth moments and then expectations. We then estimate individually the terms emerging. We consider first the space difference. From the Green's function representation and the estimates already obtained in Lemma 4 for the error terms $E^{(i)}$ we have

$$
\begin{aligned}
(2.18)\qquad & E(|\hat{A}_c(\xi_t)(z) - \hat{A}_c(\xi_t)(y)|^p) \\
& \leqq C(f_0, \lambda, \theta_c, p, T) e_{\lambda_p}(z) n^{-p/16} \\
& \quad + C(p) E\left(\left|\int_0^t (v_s, A_c(\xi_s)(\psi_{t-s}^z - \psi_{t-s}^y))\, ds\right|^p\right) \\
& \quad + C(p) E(|Z_t(\psi_{t-\cdot}^z - \psi_{t-\cdot}^y)|^p) \\
& \leqq C(f_0, \lambda, \theta_c, p, T) e_{\lambda_p}(z) n^{-p/16} \\
& \quad + C(p) E\left(\left|\int_0^t (v_s, A_c(\xi_s)(\psi_{t-s}^z - \psi_{t-s}^y))\, ds\right|^p\right) \\
& \quad + C(p) E\left(\left(\int_0^t (v_s, (\psi_{t-s}^z - \psi_{t-s}^y)^2)\, ds\right)^{p/2}\right).
\end{aligned}
$$

We break the first expectation on the right-hand side of (2.18) into two parts. Set $\delta = (|z-y|^{1/4} \vee n^{-1/2}) \wedge t$. If we restrict the integral to $[0, t-\delta)$ we obtain a part bounded by

$$
\begin{aligned}
(2.19)\quad & C(p) E\left(\left(\int_0^t (v_s, A_c(\xi_s) e_{-\lambda})\, ds\right)^p\right) \sup(\|\psi_s^z - \psi_s^y\|_\lambda^p \colon s \in (\delta, t]) \\
& \leqq C(p) E\left(\left(\int_0^t (A_c^2(\xi_s), e_{-\lambda})\, ds\right)^p\right) \\
& \qquad \times \sup(\|\psi_s^z - \psi_s^y\|_\lambda^p \colon s \in (\delta, t]) \quad \text{(Lemma 5(c))} \\
& \leqq C(\lambda, p, T) \int_0^t \|E(A_c^{2p})(\xi_s)\|_{-\lambda/2} (e_{\lambda/2}, e_{-\lambda})\, ds \\
& \qquad \times \sup(\|\psi_s^z - \psi_x^y\|_\lambda^p \colon s \in (\delta, t]) \\
& \leqq C(\lambda, p, T) e_{\lambda_p}(z)(|z-y|^{p/2} \delta^{-p} + n^{-p/2} \delta^{-3p/4}) I(\delta < t) \\
& \qquad \times \int_0^t \|E(A_c^{2p})(\xi_s)\|_{-\lambda/2} (e_{\lambda/2}, e_{-\lambda})\, ds \quad \text{(Lemma 3(e))}
\end{aligned}
$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)e_{\lambda_p}(z)(|z-y|^{p/2}\delta^{-p}+n^{-p/2}\delta^{-3p/4})$$
$$\times I(\delta<t). \quad \text{(Lemma 4(c))}$$

The integral over $[t-\delta,t]$ gives a part bounded by

$$(2.20)\quad C(p)E\left(\left(\int_{t-\delta}^{t}\|\psi_{t-s}^z+\psi_{t-s}^y\|_\lambda(v_s,A_c(\xi_s)e_{-\lambda})\,ds\right)^p\right)$$
$$\leqq C(p,T)e_{\lambda_p}(z)E\left(\left(\int_{t-\delta}^{t}(t-s)^{-2/3}(A_c^2(\xi_s),e_{-\lambda})\,ds\right)^p\right)$$
$$\text{(Lemmas 5(c) and 3(c))}$$
$$\leqq C(p,T)\delta^{(1/3)((p/2)-1)}e_{\lambda_p}(z)\int_{t-\delta}^{t}(t-s)^{-2/3}(E(A_c^p(\xi_s)),e_\lambda)\,ds$$
$$\leqq C(f_0,\lambda,\theta_c,p,T)\delta^{p/3}e_{\lambda_p}(z)\,. \quad \text{(Lemma 4(c))}$$

We break the second expectation on the right-hand side of (2.18) also into two parts. The integral over $[0,t-\delta)$ is bounded by

$$(2.21)\qquad C(p,T)E\left(\sup_{s\leqq t}(v_s,e_{-2\lambda})^{p/2}\right)\sup(\|\psi_s^z-\psi_s^y\|_\lambda^p : s\in(\delta,t])$$
$$\leqq C(f_0,\lambda,\theta_c,p,T)e_{\lambda_p}(z)(|z-y|^{p/2}\delta^{-p}+n^{-p/2}\delta^{-3p/4})$$
$$\times I(\delta<t)\quad \text{(Lemmas 4(a) and 3(e)).}$$

The integral over $[t-\delta,t]$ is bounded by

$$(2.22)\quad C(p)E\left(\left(\int_{t-\delta}^{t}\|\psi_{t-s}^z+\psi_{t-s}^y\|_0(v_s,\psi_{t-s}^z+\psi_{t-s}^y)\,ds\right)^{p/2}\right)$$
$$\leqq C(p)E\left(\left(\int_{t-\delta}^{t}(t-s)^{-2/3}(A_c(\xi_s),\bar\psi_{t-s}^z+\bar\psi_{t-s}^y)\,ds\right)^{p/2}\right)$$
$$\text{(Lemmas 5(a) and 3(c))}$$
$$\leqq C(p,T)\delta^{(1/3)((p/2)-1)}\int_{t-\delta}^{t}(t-s)^{-2/3}$$
$$\times(E(A_c^{p/2}(\xi_s)),\bar\psi_{t-s}^z+\bar\psi_{t-s}^y)\,ds$$
$$\leqq C(f_0,\lambda,\theta_c,p,T)\delta^{p/6}e_{\lambda_p}(z)\quad \text{(Lemmas 4(c) and 3(b))}\,.$$

Combining the four parts and using the definition of δ leads to

$$E(|\hat A_c(\xi_t)(z)-\hat A_c(\xi_t)(y)|^p)\leqq C(f_0,\lambda,p,T)e_{\lambda_p}(z)(|z-y|^{p/24}+n^{-p/12})\,.$$

For the time differences we have again, by subtracting the two Green's function representations,

$$(2.23)\qquad E(|\hat{A}_c(\xi_t)(z)-\hat{A}_c(\xi_s)(z)|^p)$$
$$\leqq C(f_0,\lambda,\theta_c,p,T)e_{\lambda_p}(z)n^{-p/16}$$
$$+C(p)E\left(\left(\int_s^t(v_r,A_c(\xi_r)\psi^z_{t-r})\,dr\right)^p\right)$$
$$+C(p)E\left(\left|\int_0^s(v_r,A_c(\xi_r)(\psi^z_{t-r}-\psi^z_{s-r}))\,dr\right|^p\right)$$
$$+C(p)E\left(\left(\int_s^t(v_r,(\psi^z_{t-r})^2)\,dr\right)^{p/2}\right)$$
$$+C(p)E\left(\left(\int_0^s(v_r,(\psi^z_{t-r}-\psi^z_{s-r})^2)\,dr\right)^{p/2}\right).$$

The first expectation in the right-hand side of (2.23) is bounded by

$$C(p)\sup_{r\leqq t}\|\psi^z_r\|^p_\lambda E\left(\left(\int_s^t(v_r,A_c(\xi_r)e_{-\lambda})\,dr\right)^p\right)$$
$$\leqq C(p,\lambda,T)e_{\lambda_p}(z)E\left(\left(\int_s^t(A^2_c(\xi_r),e_{-\lambda})\,dr\right)^p\right)\quad\text{(Lemmas 3(c) and 5(c))}$$
$$\leqq C(p,\lambda,T)e_{\lambda_p}(z)|t-s|^{p-1}\int_s^t(E(A^{2p}_c(\xi_r)),e_{-\lambda})dr$$
$$\leqq C(f_0,\lambda,\theta_c,p,T)e_{\lambda_p}(z)|t-s|^p\quad\text{(Lemma 4(c))}.$$

The third expectation in the right-hand side of (2.23) is bounded by

$$C(p,T)E\left(\left(\int_s^t(t-r)^{-2/3}(v_r,\psi^z_{t-r})\,dr\right)^{p/2}\right)\quad\text{(Lemma 3(c))}$$
$$=C(p,T)E\left(\left(\int_s^t(t-r)^{-2/3}(A(\xi_r),\bar{\psi}^z_{t-r})\,dr\right)^{p/2}\right)\quad\text{Lemma 5(a))}$$
$$\leqq C(p,T)|t-s|^{(1/3)((p/2)-1)}\int_s^t(t-r)^{-2/3}(E(A^{p/2}(\xi_r)),\bar{\psi}^z_{t-r})\,dr$$
$$\leqq C(f_0,\lambda,\theta_c,p,T)|t-s|^{p/6}e_{\lambda_p}(z)\quad\text{(Lemmas 3(b) and 4(c))}.$$

We break the second expectation in the right-hand side of (2.23) into two parts. Set $\bar{\delta}=(|t-s|^{1/4}\vee n^{-1/2})\wedge s$. If we restrict the integral to $[0,s-\bar{\delta})$ we obtain a part bounded, arguing as in (2.19) but using Lemma 3(f) in place of Lemma 3(e), by

$$C(p)\sup(\|\psi^z_{t-r}-\psi^z_{s-r}\|^p_\lambda\colon r\in[0,s-\bar\delta))E\left(\left(\int_0^s (v_r, A_c(\xi_r)e_{-\lambda})\,dr\right)^p\right)$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)e_{\lambda_p}(z)(|t-s|^{p/2}\bar\delta^{-3p/2}+n^{-p/2}\bar\delta^{-3p/4})I(\bar\delta<s)\,.$$

The integral over $[s-\bar\delta, s]$ gives a part bounded, arguing as in (2.20), by

$$C(p,T)e_{\lambda_p}(z)E\left(\left(\int_{s-\bar\delta}^s (s-r)^{-2/3}(v_r, A_c(\xi_r)e_{-\lambda})\,dr\right)^p\right)$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)\bar\delta^{p/3}e_{\lambda_p}(z)\,.$$

We break the fourth expectation on the right-hand side of (2.23) also into two parts. The integral over $[0,s-\bar\delta)$ is bounded, arguing as in (2.21) but using Lemma 3(f) in place of Lemma 3(e), by

$$C(p,T)\sup(\|\psi^z_{t-r}-\psi^z_{s-r}\|^p_\lambda\colon r\in[0,s-\bar\delta))E\left(\sup_{r\leqq s}(v_r,e_{-2\lambda})^{p/2}\right)$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)e_{\lambda_p}(z)(|t-s|^{p/2}\bar\delta^{-3p/2}+n^{-p/2}\bar\delta^{-3p/4})I(\bar\delta<s)\,.$$

The integral over $[s-\bar\delta, s]$ is bounded, arguing as in (2.22), by

$$C(p,T)E\left(\left(\int_{s-\bar\delta}^s (s-r)^{-2/3}(v_r,\psi^z_{t-r}+\psi^z_{s-r})\,dr\right)^{p/2}\right)$$

$$\leqq C(p,T)\bar\delta^{(1/3)((p/2)-1)}\int_{s-\bar\delta}^s (E(A_c^{p/2}(\xi_r),\bar\psi^z_{t-r}+\bar\psi^z_{s-r})\,dr$$

$$\leqq C(f_0,\lambda,\theta_c,p,T)\bar\delta^{p/6}e_{\lambda_p}(z)\,.$$

Combining the six parts and using the definition of $\bar\delta$ leads to

$$E(|\hat A_c(\xi_t)(z)-\hat A_c(\xi_s)(z)|^p)\leqq C(f_0,\lambda,\theta_c,p,T)e_{\lambda_p}(z)(|t-s|^{p/24}+n^{-p/12})$$

completing the proof of Lemma. □

We now show that these moment estimates imply tightness of the approximate densities. Define $\tilde A_c(\xi_t)(z)=\hat A(\xi_t)(z)$ on the grid $z\in n^{-2}\mathbb{Z}$, $t\in n^{-3}\mathbb{N}$. Linearly interpolate first in x and then in t to obtain a continuous $\mathscr{C}$ valued process. The next lemma shows that $\tilde A_c(\xi_t)$ and $\hat A_c(\xi_t)$ remain close. The advantage of using $\tilde A_c(\xi_t)$ is that it is continuous and it is a straightforward exercise to use the above moment estimates and the argument of Kolmogorov's continuity criterion to show that tightness holds for the processes $(\tilde A_c(\xi_t)\colon t\in[0,T])_{n=1,2,\ldots}$ as continuous $\mathscr{C}$ valued processes. Then tightness of $(A_c(\xi_t)\colon t\in[0,T])_{n=1,2,\ldots}$ as cadlag $\mathscr{C}$ valued processes and also the continuity of all limit points follow from the next lemma.

Lemma 7 *For any* $\lambda>0, T<\infty$

(a) $P(\sup_{t \leqq T} \|\tilde{A}_c(\xi_t) - \hat{A}_c(\xi_t)\|_{-\lambda} \geqq 7n^{-1/4}) \to 0$ *as* $n \to \infty$,

(b) $\sup_{t \leqq T} \|(v_0, \psi_t^{\cdot}) - P_{t/3} f_0\|_{-\lambda} \to 0$ *as* $n \to \infty$.

Proof. a) For $0 \leqq s \leqq t$ we have

$$\|(v_0, \psi_t^{\cdot}) - (v_0, \psi_s^{\cdot})\|_{-\lambda} \leqq n\|(|\psi_t^{\cdot} - \psi_s^{\cdot}|, 1)\|_0 \leqq 6n^2|t-s|$$

from Lemma 3(d). So this changes only by $O(n^{-1})$ between the grid points in $n^{-3}\mathbb{N}$. Note that the value of $A_c(\xi_t)(x)$ changes only at jump times of $P_t(x)$ or $P_t(y,x)$ for some $y \sim x$ and that each jump is bounded by $Cn^{-1/2}$. Then for $k \in \mathbb{Z}$, writing $\mathscr{P}(a)$ for a Poisson variable with mean a,

$$\begin{aligned}
&P(\exists z \in n^{-2}\mathbb{Z} \cap (k, k+1], \exists t \leqq T \text{ with } |\tilde{A}_c(\xi_t) - \hat{A}_c(\xi_t)| \geqq 7n^{-1/4} e^{\lambda(|k|-1)}) \\
&\leqq P(\exists z \in n^{-2}\mathbb{Z} \cap (k, k+1], \exists t \in n^{-2}\mathbb{N} \cap [0, T], \exists s \in [t, t+n^{-3}] \text{ with} \\
&|A_c(\xi_s)(z) - A_c(\xi_t)(z)| \vee |A_c(\xi_s)(z) - A_c(\xi_{t+n^{-2}})(z)| \geqq n^{-1/4} e^{\lambda(|k|-1)}) \\
&\leqq n^4 P\left(Cn^{-1/2}(P_{n^{-3}}(0) + \sum_{y \sim 0} P_{n^{-3}}(0, y)) \geqq n^{-1/4} e^{\lambda(|k|-1)}\right) \\
&\leqq n^4 P(C(\mathscr{P}(2n^{-2} + \theta_c n^{-3}))^p \geqq n^{p/4} e^{\lambda p(|k|-1)}).
\end{aligned}$$

Applying Chebychev (for $p \geqq 17$) and summing over k proves part (a). The proof of part (b) is delayed until Sect. 4. □

Step 4 Characterising limit points

Radon measures $(\mu k)_{k=1,2\ldots}$ on $\mathbb{R}$ converge vaguely to another measure μ if $\int \phi(z)\mu_n(dz) \to \int \phi(z)\mu(dz)$ for all continuous ϕ: $\mathbb{R} \to \mathbb{R}$ with compact support. Taking such a ϕ we have

$$(v_t^n, \phi) = (A_c(\xi_t^n), \phi) + E_t^{(3)}(\phi)$$

where $E(\sup_{t \leqq T} |E_t^{(3)}(\phi)|) \leqq C(f_0, \lambda, \theta_c, T)\|D(\phi, n^{-1/2})\|_\lambda$ by Lemma 5(b). Tightness of $(A_c(\xi_t^n)$: $t \geqq 0)$ then implies the tightness of the cadlag real valued processes $((v_t^n, \phi)$: $t \geqq 0)$. This in turn implies the tightness of $(v_t^n$: $t \geqq 0)$ as cadlag Radon measure valued processes with the vague topology once a compact containment condition is checked [3, Theorem 3.6.4]. Proposition 4(a) implies this compact containment condition. Note that all limit points are again continuous.

We now fix a convergent subsequence for the pair $(A_c(\xi_t^n), v_t^n))_{t \geqq 0}$. By a theorem of Skorokhod (a slight extension of [6, Theorem 2.1.8]) we may find variables with the same distribution as $(\xi_t^n(z)$: $t \geqq 0, z \in n^{-2}\mathbb{Z}, n \geqq 0)$ for which the convergence is almost sure. Since we are only interested in identifying the distribution of the limit there is no danger in continuing to label this almost sure convergent subsequence as $(A_c(\xi_t^n), v_t^n))_{t \geqq 0}$. Also, since the limits are continuous, the almost sure convergence holds not only in the Skorokhod topology but in the topology of uniform convergence on compacts in $[0, \infty]$ ([6, Lemma 2.10.1]). Thus we have that with probability one, for all $T < \infty$, $\lambda > 0$, ϕ of compact support

$$\sup_{t \leqq T} \|A_c(\xi_t^n) - u_t\|_{-\lambda} \to 0 \quad \text{as } n \to \infty\,,$$

$$\sup_{t \leqq T} |\int \phi(x) v_t^n(dx) - \int \phi(x) v_t(dx)| \to 0 \quad \text{as } n \to \infty\,.$$

It is easy to see that $v_t(dx) = u_t(x)\,dx$ for all $t \geqq 0$.
Take ϕ three times continuously differentiable and with compact support. From (2.9) with $\phi_t = \phi$ we have that

$$\begin{aligned}(2.24)\qquad Z_t(\phi) &= \int \phi(x) v_t^n(dx) - \int \phi(x) v_0^n(dx)\\ &\quad - \int_0^t \int (\theta_c \phi(x) + A_c(\phi)(x)) v_s^n(dx)\,ds\\ &\quad + (1 + \theta_c n^{-1}) \int_0^t \int \phi(x) A_c(\xi_s^n)(x) v_s^n(dx)\,ds\\ &\quad - E_t^{(1)}(\phi) + E_t^{(2)}(\phi)\end{aligned}$$

is a martingale. From (2.6) and (2.8) and Burkholder's inequality

$$\begin{aligned}&E\left(\sup_{t \leqq T} |E_t^{(1)}(\phi)|^2 + |E_t^{(2)}(\phi)|^2\right)\\ &\quad \leqq C(f_0, \lambda, \theta_c, T)(\|D(\phi, n^{-1/2})\|_\lambda^2 + n^{-1/2}\|\phi\|_\lambda^2) + Cn^{-1}\,.\end{aligned}$$

Dropping to a further subsequence if necessary the error terms then converge to zero for all t almost surely. Taylor's theorem shows that, when linearly interpolated, $A_c(\phi) \to (1/6)\Delta\phi$ as $n \to \infty$ uniformly on the support of ϕ. If measures $\mu_n \to \mu$ vaguely and functions f_n are all supported on one compact and $f_n \to f$ uniformly then $\int f_n(x)\mu_n(dx) \to \int f(x)\mu(dx)$. Using this all the terms on the right-hand side of (2.24) converge for all $t \geqq 0$, almost surely. Hence the left-hand side also converges to a local martingale $z_t(\phi)$ where

$$\begin{aligned}(2.25)\quad z_t(\phi) &= \int \phi(x) u_t(x)\,dx - \int \phi(x) u_0(x)\,dx\\ &\quad - \int_0^t \int (\theta_c \phi(x) + (1/6)\Delta(\phi)(x)) u_s(x) - \phi(x) u_s^2(x)\,dx\,ds\,.\end{aligned}$$

Note that since the right-hand side is continuous so is $z_t(\phi)$. Also from (2.5)

$$Z_t^2(\phi) - (2 + \theta_c n^{-1}) \int_0^t \int \phi^2(x) v_s^n(dx)\,ds$$

is a martingale. Letting $n \to \infty$ we have almost surely, for all $t \geqq 0$

$$(2.26)\qquad z_t^2(\phi) - 2\int_0^t \int \phi^2(x) u_s^2(x)\,dx\,ds$$

is a continuous local martingale. Equations (2.25) and (2.26) now hold simultaneously for a countable collection of such test functions (ϕ_n). We may choose (ϕ_n) so that for any twice continuously differentiable function $\psi\colon \mathbb{R} \to \mathbb{R}$ of compact support, there is a subsequence $(\phi_{n(k)})$ so that $\phi_{n(k)} \to \psi$ and $\Delta\phi_{n(k)} \to \Delta\psi$ uniformly. Using this we find that (2.25) and (2.26) hold for all such ψ and thus that u_t solves the martingale problem associated with the stochastic p.d.e. (1.1). It is now straightforward to show that, with respect to some white noise, u_t is actually a solution to (1.1) (see [11], V20 for the similar argument in the case of stochastic o.d.e's). Thus all limit points have the same law and the convergence in Theorem 1 is proved.

3 Long range voter process

We risk a little confusion by redefining some of our notation. Although the scaling is different, most of the steps for the voter process are the same as for the contact process and we feel that to use different notation would obscure the similarities. Indeed once the approximate martingale problem is established almost all the remaining steps are left to the reader.

For f, g: $n^{-1}\mathbb{Z} \to \mathbb{R}$ we write (f, g) for $n^{-1}\sum_x f(x)g(x)$ and if v is a measure of $n^{-1}\mathbb{Z}$ we write (v, f) for $\int f\,dv$. Again $\| f \|_\lambda = \sup\{|f(x)e_\lambda(x)|: x \in n^{-1}\mathbb{Z}\}$ and for $x \in n^{-1}\mathbb{Z}, \delta > 0$

$$D(f, \delta)(x) = \sup\{|f(y) - f(x)|: |y - x| \leqq \delta,\ y \in n^{-1}\mathbb{Z}\},$$

$$\Delta_v(f)(x) = n^{1/2} \sum_{y \sim x} (f(y) - f(x)).$$

The graphical construction uses independent families of i.i.d. Poisson processes:

$$(P_t(x, y):\ x, y \in n^{-1}\mathbb{Z},\ x \sim y) \quad \text{i.i.d. Poisson processes of rate } n^{1/2},$$

$$(\bar{P}_t(x, y):\ x, y \in n^{-1}\mathbb{Z},\ x \sim y) \quad \text{i.i.d. Poisson processes of rate } \theta_c n^{-1/2}.$$

At a jump time of $P_t(x, y)$ the voter at x adopts the opinion of the voter at y. At a jump of $\bar{P}_t(x, y)$ the voter at x adopts the opinion of the voter at y provided that it is the opinion 1.

We label the opinion of the voter at site x at time t by $\xi^n_t(x)$. Define $v^n_t = n^{-1}\sum_x \delta_x I(\xi^n_t(x) = 1)$ so that $(\xi^n_t, \phi) = (v^n_t, \phi)$. We now derive the approximate martingale problem. We again drop the superscript to simplify notation. The dynamics of the voter model are captured in the equation

$$\begin{aligned}\xi_t(x) =& \xi_0(x) + \sum_{y \sim x} \int_0^t (\xi_{s-}(y) - \xi_{s-}(x))\, dP_s(x, y) \\ &+ \sum_{y \sim x} \int_0^t (1 - \xi_{s-}(x))\xi_{s-}(y)\, d\tilde{P}_s(x, y).\end{aligned}$$

Take a test function ϕ: $[0, \infty) \times n^{-1}\mathbb{Z} \to \mathbb{R}$ with $t \to \phi_t(x)$ continuously differentiable and satisfying (2.1). Apply integration by parts to $\xi_t(x)\phi_t(x)$ and sum over x to obtain, for $t \leqq T$

$$\begin{aligned}(v_t, \phi_t) =& (v_0, \phi_0) + \int_0^t (v_s, \partial_s \phi_s)\, ds \\ &+ n^{-1} \sum_x \sum_{y \sim x} \int_0^t (\xi_{s-}(y) - \xi_{s-}(x))\phi_s(x)\, dP_s(x, y)\end{aligned}$$

$$+ n^{-1} \sum_x \sum_{y \sim x} \int_0^t (1 - \xi_{s-}(x))\xi_{s-}(y)\phi_x(x)\, d\bar{P}_s(x,y)$$

$$= (v_0, \phi_0) + \int_0^t (v_s, \partial_s \phi_s)\, ds$$

$$(3.1) \qquad + n^{-1} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(y)(\phi_s(x) - \phi_s(y))\, dP_s(x,y)$$

$$(3.2) \qquad + n^{-1} \sum_x \sum_{y \sim x} \int_0^t (1 - \xi_{s-}(x))\xi_s(y)\phi_s(x)\, d\bar{P}_s(x,y)$$

$$(3.3) \qquad + n^{-1} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(x)\phi_s(x)(dP_s(y,x) - dP_s(x,y))\,.$$

We break term (3.1) into two parts, an average term and a fluctuation term:

$$n^{-1} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(y)(\phi_s(x) - \phi_s(y)) n^{1/2}\, ds$$

$$+ n^{-1} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(y)(\phi_s(x) - \phi_s(y))(dP_s(x,y) - n^{1/2} ds)$$

$$= \int_0^t (v_s, \Delta_v \phi_s) ds + E_t^{(3)}(\phi)$$

where

$$E_t^{(3)}(\phi) := n^{-1} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(y)(\phi_s(x) - \phi_s(y))(dP_s(x,y) - d\langle P(x,y)\rangle_s)$$

is a martingale with predictable brackets process given by

$$d\langle E^{(3)}(\phi)\rangle_t = n^{-2} \sum_x \sum_{y \sim x} \xi_t(y)(\phi_t(x) - \phi_t(y))^2 n^{1/2} dt$$

$$\leqq 2c_2(D^2(\phi_t, n^{-1/2}), 1)\, dt$$

$$\leqq 2c_2 \|D(\phi_t, n^{-1/2})\|_\lambda^2 (1, e_{-2\lambda})\, dt\,.$$

Alternatively we may bound

$$d\langle E^{(3)}(\phi)\rangle_t \leqq 4c_2 \|\phi_t\|_0 (\phi_t, 1)\,.$$

We break term (3.2) into two parts, an average term and a fluctuation term:

$$n^{-1} \sum_x \sum_{y \sim x} \int_0^t (1 - \xi_{s-}(x))\xi_{s-}(y)\phi_s(x)\theta_v n^{-1/2} ds)$$

$$+ n^{-1} \sum_x \sum_{y \sim x} \int_0^t (1 - \xi_{s-}(x))\xi_{s-}(y)\phi_s(x)(d\bar{P}_s(x,\, y) - \theta_v n^{-1/2} ds)$$

$$= 2c_2\theta_v \int_0^t (A_v(\xi_s), \phi_s) - (v_s, A_v(\xi_s)\phi_s)\, ds + E_t^{(4)}(\phi)$$

where

$$E_t^{(4)}(\phi) := n^{-1}\sum_x \sum_{y\sim x} \int_0^t (1-\xi_{s-}(x))\xi_{s-}(y)\phi_s(x)(d\bar{P}_s(x,y) - d\langle \bar{P}(x,y)\rangle_s)$$

is a martingale with predictable brackets process given by

$$\langle E^{(4)}(\phi)\rangle_t \leqq n^{-2}\sum_x \sum_{y\sim x} \int_0^t \phi_s^2(x)\theta_v n^{-1/2} ds$$

$$\leqq 4\theta_v n^{-1} \int_0^t \|\phi_s\|_\lambda^2 (e_{-2\lambda}, 1)\, ds\,.$$

The last term (3.3) is a martingale which we write as $Z_t(\phi) = n^{-1}\sum_x Z_t^x(\phi)$ where

$$Z_t^x(\phi) := \sum_{y\sim x} \int_0^t \xi_{s-}(x)\phi_s(x)(dP_s(y,x) - dP_s(x,y))\,.$$

From the independence of $(P_t(x, y)\colon\ x, y \in n^{-1}\mathbb{Z})$ we conclude that

$$\langle (P_t(x,y) - n^{1/2}t,\ P_t(x',y') - n^{1/2}t\rangle = n^{1/2}t\, I(x = x',\ y = y')\,.$$

Therefore

$$\begin{aligned}
&\langle Z^x(\phi), Z^{x'}(\phi)\rangle_t \\
&= \sum_{y\sim x} \sum_{y'\sim x'} \int_0^t \xi_{s-}(x)\xi_{s-}(x')\phi_s(x)\phi_s(x') \\
&\qquad \times 2n^{1/2}(I(y = y', x = x') - I(x = y',\ y = x'))ds \\
&= 4nc_2 I(x = x') \int_o^t \xi_s(x)\phi_s^2(x)ds \\
&\qquad - 2n^{1/2} I(x \sim x') \int_0^t \xi_s(x)\xi_s(x')\phi_s(x)\phi_s(x')ds\,.
\end{aligned}$$

Then

$$\begin{aligned}
\text{(3.4)} \qquad \langle Z(\phi)\rangle_t &= n^{-2}\sum_x \sum_{x'} \langle Z^x(\phi), Z^{x'}(\phi)\rangle_t \\
&= 4c_2 \int_0^t (\xi_s, \phi_s^2) - (\xi_s\phi_s, A_c(\xi_s\phi_s))ds \\
&\leqq 4c_2 \int_0^t \|\phi_s\|_\lambda^2(e_{-2\lambda}, 1)ds\,.
\end{aligned}$$

Collecting terms we have the following approximate martingale decomposition

$$(v_t, \phi_t) = (v_0, \phi_0) + \int_0^t (v_s, \partial_s\phi_s + \Delta_v\phi_s)ds$$

$$+ 2c_2\theta_v \int_0^t (A_v(\xi_s), \phi_s) - (v_s, A_v(\xi_s)\phi_s)ds + Z_t(\phi)$$

$$+ E_t^{(3)}(\phi) + E_t^{(4)}(\phi) .$$

With this approximate decomposition established the proof now follows that for the contact process. Note that since $A_v(\xi_t) \in [0, 1]$ by definition, the moments estimates in Lemma 4(a) and (c) are unnecessary.

We point out only one difference in deriving the limiting stochastic p.d.e. As for the contact process it is established that $(v_t^n, A_c(\xi_t^n))$ are tight and we write $(u_t(x)dx, u_t)$ for a limit point. Then for ϕ three times continuously differentiable and of compact support, $Z_t(\phi)$ converges to a continuous martingale $z_t(\phi)$ as $n \to \infty$ where

(3.5)
$$z_t(\phi) = \int \phi(x)u_t(x)dx - \int \phi(x)u_o(x)dx$$

$$- \int_0^t \int (\theta_c\phi(x) + (1/6)\Delta(\phi)(x))u_s(x)$$

$$+ 2\theta_v\phi(x)u_s(x)(1 - u_s(x))dx\, ds .$$

From (3.4)

(3.6)
$$Z_t^2(\phi) - 4c_2 \int_0^t (\xi_s^n, \phi^2) - (\xi_s^n\phi, A_v(\xi_s^n\phi))ds$$

$$= Z_t^2(\phi) - 4c_2 \int_0^t (\xi_s^n(1 - A_v(\xi_s^n)), \phi^2)ds$$

$$+ 4c_2 \int_0^t (\xi_s^n\phi, A_v(\xi_s^n\phi) - A_v(\xi_s^n)\phi)ds$$

$$= Z_t^2(\phi) - 4c_2 \int_0^t \int (1 - A_v(\xi_s^n)(x))\phi^2(x)v_s^n(dx)ds + E_t^{(5)}(\phi) .$$

This defines a third error term $E^{(5)}$ for which

$$|E_t^{(5)}(\phi)| \leqq 4c_2 \int_0^t n^{-1} \sum_x \xi_s^n(x)\phi(x)(2c_2n^{1/2})^{-1} \sum_{y\sim x} \xi_s^n(y)|\phi(y) - \phi(x)|ds$$

$$\leqq 4c_2t(\phi, D(\phi, n^{-1/2})) \to 0 .$$

All terms in (3.6) again converge showing that

(3.7)
$$z_t^2(\phi) - \int_0^t \int (1 - u_s(x))u_s(x)\phi^2(x)dx\, ds$$

is a continuous martingale. Equations (3.5) and (3.7) together imply that u_t is a solution of the stochastic p.d.e. (1.2) with respect to some white noise.

4 Estimates from the local limit theorem

The purpose of this section is to prove Lemmas 3 and 7(b). We need an error bound in the local limit theorem. Since our distribution changes with n we have not been able to plug directly into a result from the literature but, as we see below, the usual characteristic function proof gives a suitable error bound.

Let $(Y_i)_{1=1,2\ldots}$ be i.i.d. and uniformly distributed on $(jn^{-2}\colon\ |j| \leqq n^{3/2})$. Set $\rho(t) = E(\exp(itY_1))$ and $S_k = \sum_{i=1}^k Y_i$. Note that $E(Y_1^2) = c_3/3n$, $E(Y_1^4) = c_4/5n^2$ where $c_3(n), c_4(n) \to 1$ as $n \to \infty$.

The relation between the test functions ψ_t^z, $\bar{\psi}_t^z$ defined in (2.10) and S_k is

$$(4.1)\qquad \psi_t^z(x) = \sum_{k=0}^{\infty} \exp(-(n+\theta_c)t)((n+\theta_c)t)^k(k!)^{-1}n^2P(S_{k+1} = x - z)\,,$$

$$(4.2)\qquad \bar{\psi}_t^z(x) = \sum_{k=0}^{\infty} \exp(-(n+\theta_c)t)((n+\theta_c)t)^k(k!)^{-1}n^2P(S_k = x - z)\,.$$

We let $(\mathscr{P}_t)_t$ be a Poisson process with rate $(n+\theta_c)$. We shall use the bound $E((\mathscr{P}_t+1)^a) \leqq C(a)(nt)^a$ for all $a < 0$. The following lemma and corollary give an error bound in the local limit theorem for S_k.

Lemma 8 *There exists $n_0 < \infty$ such that for all $n \geqq n_0$, $k \geqq 1$*

(a) $|\rho^k(t) - \exp(-c_3kt^2/6n)| \leqq Ck^{-1}\exp(-c_3kt^2/12n)$ *for* $t \leqq (n/3)^{1/2}$,

(b) $|\rho(t)| \leqq \exp(-c_3t^2/12n)$ *for* $t \leqq (2n)^{1/2}$,

(c) $|\rho(t)| \leqq 3/4$ *for* $t \in [(2n)^{1/2}, \pi n^2]$.

Proof. (a) This bound is obtained by substituting the moments of Y_1 into Theorem 8.5 from Bhattacharya and Rao [1].

(b) From Durrett [4, Chap. 2], (3.7) and the moments of X_1

$$(4.3)\qquad \rho(t) = 1 - (c_3/6n)t^2 - (c_4/120n^2)t^4e$$

where $|e| \leqq 1$. First choose n_0 such that $c_i(n) \in [1/2, 2]$ for $i = 1,\ldots,4$, $n \geqq n_0$. Then $|\rho(t)| \leqq 1 - (c_3/12n)t^2$ when $t \leqq (2n)^{1/2}$, $n \geqq n_0$ and the first bound follows from the inequality $1 - x \leqq e^{-x}$.

(c) Let $m = \sup(j \in \mathbb{N}\colon\ j \leqq n^{3/2})$. Then

$$\begin{aligned}|\rho(t)| &= \left|(2c_1n^{3/2})^{-1}\sum_{j=-m}^{m} e^{itj/n^2}\right| \\ &= |(2c_1n^{3/2})^{-1}(e^{it(m+1)/n^2} - e^{-itm/n^2})(e^{it/n^2} - 1)^{-1}| \\ &\leqq (c_1n^{3/2})^{-1}|\sin(2/n^{3/2})|^{-1} \quad \text{for } t \in [(2n)^{1/2}, \pi n^2]\,.\end{aligned}$$

The right-hand side converges to 1/2 as $n \to \infty$. We may choose n_0 to obtain the desired result.

Corollary 9 *For $n \geqq n_0$, $y \in n^{-2}\mathbb{Z}$*

$$|n^2P(S_k = y) - p(c_3k/3n, y)| \leqq C(n^2e^{-k/48} + n^{1/2}k^{-3/2}) .$$

Proof. Following a standard characteristic function proof (see [4, Sect. 2.5]) we have for $y \in n^{-2}\mathbb{Z}$, $k \geqq 1$

$$n^2P(S_k = y) = (2\pi)^{-1} \int_{-\pi n^2}^{\pi n^2} e^{ity}\rho^k(t)\,dt ,$$

$$p(c_3k/3n, y) = (2\pi)^{-1} \int e^{ity}e^{-c_3kt^2/6n}dt .$$

Subtracting these equations and using the bounds from Lemma 8 we have

$$\begin{aligned}
&|n^2P(S_k = y) - p(c_3k/3n, y)| \\
&\leqq \pi^{-1} \int_{\pi n^2}^{\infty} e^{-c_3kt^2/6n}dt + \pi^{-1} \int_{(n/3)^{1/2}}^{\pi n^2} |\rho^k(t)| + e^{-c_3kt^2/6n}dt \\
&\quad + \pi^{-1} \int_{0}^{(n/3)^{1/2}} |\rho^k(t) - e^{-c_3kt^2/6n}|dt \\
&\leqq 2\pi^{-1} \int_{(n/3)^{1/2}}^{\infty} e^{-c_3kt^2/12n}dt + \pi^{-1} \int_{(2n)^{1/2}}^{\pi n^2} (3/4)^k dt \\
&\quad + \pi^{-1} \int_{0}^{(n/3)^{1/2}} Ck^{-1} \exp(-c_3kt^2/12n)dt \\
&\leqq C(n^{1/2}k^{-1}e^{-c_3k/36} + n^2(3/4)^k + n^{1/2}k^{-3/2}) \\
&\leqq C(n^2e^{-k/48} + n^{1/2}k^{-3/2}) . \quad \square
\end{aligned}$$

Proof of Lemma 3 That $(\psi_t^z, 1) = \bar{\psi}_t^z, 1) = 1$ is immediate from (4.1) and (4.2). By induction we have that $P(S_k = x) \leqq Cn^{1/2}$ for all $x \in n^{-1}\mathbb{Z}$, $k \geqq 1$. Substituting in (4.1) and (4.2) shows that

$$\|\psi_t^z\|_0 \leqq Cn^{1/2}, \ \|\bar{\psi}_t^z\|_0 \leqq Cn^2$$

proving part (a).

We omit the easy proof of the exponential bound $E(\exp(\mu Y_1)) \leqq \exp(5\mu^2n^{-1})$. Then from (4.1), for $t \leqq T$,

$$\begin{aligned}
(\psi_t^z, e_\lambda) &= \sum_{k=0}^{\infty} \exp(-(n+\theta_c)t)((n+\theta_c)t)^k(k!)^{-1} \sum_x e_\lambda(x)P(S_{k+1} = x - z) \\
&\leqq 2e_\lambda(z) \sum_{k=0}^{\infty} \exp(-(n+\theta_c)t)((n+\theta_c)t)^k(k!)^{-1}E(\exp(\lambda S_{k+1})) \\
&\leqq 2e_\lambda(z) \sum_{k=0}^{\infty} \exp(-(n+\theta_c)t)((n+\theta_c)t)^k(k!)^{-1}\exp(5\lambda^2n^{-1}(k+1)) \\
&= 2e_\lambda(z)\exp(5\lambda^2n^{-1} + (n+\theta_c)t(e^{5\lambda^2n^{-1}} - 1)) \leqq C(\lambda, T)e_\lambda(z) .
\end{aligned}$$

A similar bound holds for $(\bar{\psi}_t^z, e_\lambda)$ proving part (b).

By induction on k one may show that the function $x \rightarrow P(S_k = x - z)$ is non-negative, symmetric and unimodal about z. The same properties then hold for $\psi_t^z, \bar{\psi}_t^z$ by substituting in (4.1) and (4.2). Using this unimodality we have, for $|x| \geqq 1$,

$$\begin{aligned} P(S_k = x) &\leqq n^{-2} P(S_k \geqq |x| - 1) \\ &\leqq n^{-2} \exp(-\mu(|x| - 1)) E(\exp(\mu S_k)) \\ &\leqq n^{-2} \exp(-\mu(|x| - 1)) \exp(5\mu^2 k n^{-1}) . \end{aligned}$$

Now substitution of this bound into (4.1) gives for any μ

$$(4.4) \qquad \psi_t^z(x) \leqq C(\mu, T) \exp(-\mu|x - z|) \quad \text{for } t \leqq T, \ |x - z| \geqq 1 .$$

From Corollary 9 and (4.1) we have

$$(4.5) \qquad \psi_t^z(x) = E(p(c_3(\mathscr{P}_t + 1)/3n, x - z)) + e(n, t, x - z)$$

where, setting $\beta = 1 - e^{-1/48}$,

$$\begin{aligned} (4.6) \qquad |e(n, t, x)| &\leqq Cn^2 E(\exp(-\mathscr{P}_t/48)) + Cn^{1/2} E((1 + \mathscr{P}_t)^{-3/2}) \\ &\leqq Cn^2 e^{-\beta n t} + Cn^{-1} t^{-3/2} \\ &\leqq Cn^{-1}(1 + t^{-3/2}) \quad \text{for } t \geqq n^{-3/4}. \end{aligned}$$

Then using the bound $|p(t, x)| \leqq Ct^{-1/2}$ we have from (4.5)

$$\begin{aligned} \psi_z^t(x) &\leqq Cn^{1/2} E((1 + \mathscr{P}_t)^{-1/2}) + |e(n, t, x - z)| \\ &\leqq Ct^{-1/2} + Cn^{-1}(1 + t^{-3/2}) \quad \text{for } t \geqq n^{-3/4} \\ &\leqq C(T) t^{-2/3} \ \text{for } t \in [n^{-3/4}, T] . \end{aligned}$$

Combining this bound with the bound in part (a) and (4.4) leads to the bound

$$\psi_t^z(x) \leqq C(\lambda, T)(n^{1/2} \wedge t^{-2/3}) \exp(-\lambda|x - z|)$$

which implies part (c). For part (d) we differentiate (4.1), giving

$$\begin{aligned} &(|\psi_t^z - \psi_s^z|, 1) \\ &\leqq \int_s^t \sum_{k=0}^{\infty} \exp(-(n + \theta_c)q)((n + \theta_c)q)^k (k!)^{-1} \\ &\qquad \times | -(n + \theta_c) + kq^{-1}| \sum_x P(S_{k+1} = z - x) \, dq \\ &\leqq 2n \int_s^t \sum_{k=0}^{\infty} \exp(-(n + \theta_c)q)((n + \theta_c)q)^k (k!)^{-1} (1 + kq^{-1} n^{-1}) \, dq \\ &\leqq 6n \int_s^t dq . \end{aligned}$$

For parts (e)–(g) we fix $n^{-3/4} \leqq t \leqq T$, $z, y \in n^{-2}Z$, $|z - y| \leqq 1$. We have from (4.5) and the inequality $|p(t,x) - p(t,y)| \leqq Ct^{-1}|x-y|$

$$\|\psi_t^z - \psi_t^y\|_0 \leqq C|z-y|nE((\mathscr{P}_t+1)^{-1}) + 2\|e(n,t)\|_0$$
$$\leqq C(T)(|z-y|t^{-1} + n^{-1}t^{-3/2}) \quad \text{(using (4.6))}.$$

But from (4.4) we have $\psi_t^z(x) + \psi_t^y(x) \leqq C(\lambda, T)\exp(-2\lambda|x-z|)$ for $|x-z| \geqq 2$. So

$$\|\psi_t^z - \psi_t^y\|_\lambda \leqq \sup_{x:|x-z|<2} C(\lambda)\|\psi_t^z - \psi_t^y\|_0 e_\lambda(x)$$
$$+ \sup_{x:|x-z|\geqq 2} \min(\|\psi_t^z - \psi_t^y\|_0, C(\lambda, T)\exp(-2\lambda|x-z|))e_\lambda(x)$$
$$\leqq C(\lambda, T)e_\lambda(z)\left(\|\psi_t^z - \psi_t^y\|_0 + \|\psi_t^z - \psi_t^y\|_0^{1/2}\right)$$
$$\leqq C(\lambda, T)e_\lambda(z)(|z-y|^{1/2}t^{-1} + n^{-1/2}t^{-3/4}).$$

For part (f) we argue exactly as in part (e) but use the inequality $|p(t,x) - p(s,x)| \leqq C|t-s|s^{-3/2}$ for $0 \leqq s \leqq t$. Finally part (g) follows from part (e) and the definition of $D(\psi_t^z, n^{-1/2})$. □

Proof of Lemma 7(*b*) Write $f_z(x) = f_0(z+x)$. Then

$$\|(v_0, \psi_t^{\cdot}) - P_{t/3}f_0\|_{-\lambda} = \|(A_c(\xi_0), \bar{\psi}_t^{\cdot}) - P_{t/3}f_0\|_{-\lambda}$$
$$\leqq \|(A_c(\xi_0) - f_0, \bar{\psi}_t^{\cdot})\|_{-\lambda} + \|(f_{\cdot}, \bar{\psi}_t^0 - p(t/3))\|_{-\lambda}.$$

From Lemma 3(b)

$$\sup_{t\leqq T}\|(A_c(\xi_0) - f_0, \bar{\psi}_t^{\cdot})\|_{-\lambda} \leqq \|A_c(\xi_0) - f_0\|_{-\lambda} \sup_{t\leqq T}\|(e_\lambda, \bar{\psi}_t^{\cdot} f)\|_{-\lambda}$$
$$\leqq C(\lambda, T)\|A_c(\xi_0) - f_0\|_{-\lambda} \to 0.$$

To control $\|(f_{\cdot}, \bar{\psi}_t^0 - p_{t/3})\|_{-\lambda}$ we argue from the Skorokhod representation (see [4, Theorem 7.6.3]) for $\bar{\psi}$. We may find a Brownian motion B with $B(0) = 0$ and stopping times $0 = T_0 \leqq T_1 \leqq T_2 \leqq \cdots$ with $(T_m - T_{m-1})_m$ i.i.d. variables satisfying $E(T_1) = E(Y_1^2)$, $E(T_1^2) \leqq CE(Y_1^4)$ and $B(T_m)$ equal in distribution to S_m. We may also assume that the Poisson process $(\mathscr{P}_t)_t$ is defined on the same probability space and is independent of $B, (T_m)_m$. Then from (4.2)

$$(f_z, \bar{\psi}_t^0 - p(t/3)) = E(f_z(B(T_{\mathscr{P}_t})) - f_z(B(t/3))).$$

Fix $\varepsilon > 0$ and choose $\delta \in (0, 1]$ so that $|f_0(x) - f_0(y)| \leqq \varepsilon e_\lambda(x)$ whenever $|x-y| \leqq \delta$. Define

$$\Omega(\eta) = \{|T_{\mathscr{P}_t} - (t/3)| \leqq \eta, \ \forall t \leqq T\},$$
$$\Omega(\eta, \delta) = \{|B(t) - B(s)| \leqq \delta, \ \forall |t-s| \leqq \eta, \ s, t \leqq T\}.$$

Then for $t \leqq T$

$$(f_z, \bar{\psi}_t^0 - p(t/3))$$
$$\leqq \varepsilon E(e_\lambda(z + B(t/3))) + E((f_z(B(T_{\mathscr{P}_t}))$$
$$+ f_z(B(t/3)))I(\Omega^c(\eta) \cup \Omega^c(\eta, \delta)))$$
$$\leqq C(\lambda, T)\varepsilon e_\lambda(z) + 2(P(\Omega^c(\eta) \cup \Omega^c(\eta, \delta)))^{1/2}(f_z^2, \bar{\psi}_t^0 + p(t/3))^{1/2}$$
$$\leqq C(\lambda, T)e_\lambda(z)(\varepsilon + (P(\Omega^c(\eta) \cup \Omega^c(\eta, \delta)))^{1/2}).$$

We first choose η small so that $P(\Omega^c(\eta, \delta)) \leqq \varepsilon$. The process $T_{\mathscr{P}_t} - c_3(1 + \theta_c n^{-1})t/3$ is a martingale and using a simple martingale inequality we may then choose n large so that $P(\Omega^c(\eta)) \leqq \varepsilon$. This shows that

$$\sup_{t \leqq T} \|(f_\cdot, \bar{\psi}_t^0 - p(t/3))\|_{-\lambda} \to \quad \text{as } n \to \infty$$

which completes the proof. □

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