07/09/16

LECTURE - 6.

Recall: $|\sigma(x)-\sigma(y)| \le \lambda|x-y|$, $u_0 : \mathbb{R} \to \mathbb{R}$ bounded m'ble

$\exists\ \beta > 0$, & $u \in \mathcal{L}^{\beta,2}$ such that $u(t,x) = P_t * u_0(x)$

$+ \int_0^t \int_{\mathbb{R}} \sigma(u(s,y))\, p_{t-s}(x-y)\, \xi(dy,ds)$. ~~is a sol^n to~~

- u has a continuous modification
- (uniqueness) in the class of sol^{n}s

$$\sup_{x\in\mathbb{R}} \mathbb{E}\,|u(t,x)|^k \le L^k e^{Lk^3 t} \quad \forall\ k \in [1,\infty) \ \&\ t \ge 0.$$

Ex. 1

$$\frac{\partial u}{\partial t} = \Delta u - u + \xi \qquad 0 < x < \pi,\ t > 0$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial}{\partial x} u(t,\pi) = 0 \qquad \forall\ t > 0$$

$$u(0,x) = 0.$$

$\phi_0(x) = \frac{1}{\sqrt{\pi}}$, $\phi_k(x) = \sqrt{\frac{2}{\pi}}\cos(kx)$, $k \ge 1$.

$$\Delta\varphi_k = -k\,\varphi_k \Rightarrow \Delta\varphi_k - \varphi_k = -(k^2+1)\varphi_k \qquad \lambda_k = -(k^2+1)$$

$$P_t(x,y) = \sum_{k=0}^{\infty} \varphi_k(x)\varphi_k(y)\, e^{\lambda_k t} \quad \text{(Green's function)}$$

$$u(t,x) = \int_0^t \int_0^\pi P_{t-s}(x,y)\, \xi(dy,ds)$$

[Compare with Duhamel's principle sol^n in Lecture 4]

[whenever Green's fn exists to $\frac{\partial u}{\partial t} = \Delta u + b(u)$ the above approach might work. Mainly when $b(\cdot)$ is linear.]

$$u(t,x) = \int_0^t \int_0^\pi \sum_{k=0}^{\infty} \varphi_k(x)\varphi_k(y)\, e^{\lambda_k(t-s)}\, \xi(dy,ds)$$

$$= \sum_{k=0}^{\infty} \varphi_k(x) \underbrace{\int_0^t e^{\lambda_k(t-s)} \int_0^\pi \varphi_k(y)\, \xi(dy,ds)}_{A_k} \longrightarrow A_k(t)$$

$$= \sum_{k=0}^{\infty} \varphi_k(x) A_k(t)$$

$$B_k(t) = \int_0^t \int_0^\pi \varphi_k(y)\, \xi(dy, ds) \quad \text{- i.i.d. BM} \quad \mathrm{Cov}(B_k(t), B_l(t)) = 0.$$

independent (not identical)

$$A_k(t) = \int_0^t e^{\cancel{-}\lambda_k (t-s)}\, dB_k(s)$$

$$dA_k = \cancel{-}\lambda_k A_k\, dt + dB_k(t)$$

} - Ornstein-Uhlenbeck

$$A_k(t) \overset{d}{=} N\left(0, \frac{e^{2\lambda_k t} - 1}{2\lambda_k}\right)$$

$$A_k(0) = 0.$$

$$B_x = \frac{\xi_0}{\sqrt{3}} + \sum_{k=1}^{\infty} \frac{\xi_k}{k} \phi_k(x), \quad \xi_k \text{ i.i.d. } N(0,1) \quad 0 < x < \pi.$$

B.M.

$$u(t,x) = \frac{B_x}{\sqrt{2}} + R_x^{(t)}, \quad R_x^{(t)} = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \left[\sqrt{\frac{e^{2\lambda_k t} - 1}{\lambda_k}}\, \xi_k - \frac{\xi_k}{k} \right] \varphi_k(x)$$

$R_x^{(t)}$ is twice diff'ble in 'x'.

σ - Lipschitz.

$\boxed{\text{Ex. 2}}$

$$\frac{\partial v}{\partial t} = \Delta v - v + \sigma(v)\xi \qquad 0 < x < L \quad 0 < t < T$$

$$v(x,0) = \psi(x); \quad \frac{\partial v}{\partial x}(t,0) = \frac{\partial v}{\partial x}(t,L) = 0.$$

$$P_t(x,y) = \frac{e^{-t}}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \left[e^{-\frac{(y-x-2nL)^2}{4t}} + e^{-\frac{(y+x-2nL)^2}{4t}} \right]$$

$$\int_0^L P_s(x,y) P_t(y,z)\, dy = P_{t+s}(x,z) \qquad p_t(x,y) < \frac{C_T}{\sqrt{t}} e^{-t} e^{-\frac{|y-x|^2}{4t}}$$

$$P_t(x,y) = P_t(y,x)$$

$$v(t,x) = P_t * \psi(x) + \int_0^t \int_0^L \sigma(v(s,y))\, P_{t-s}(x,y)\, \xi(dy, ds) \qquad (*)$$

$$\sup_{x\in[0,L]} \sup_{0<t<T} \mathbb{E}|v(t,x)|^2 < M.$$

v^1, v^2 - two sol^n to $\circledast$ $\qquad u = v^1 - v^2$

Uniqueness Proof for $\circledast$ $\qquad 0 < t < T$

$$u(t,x) = \int_0^t \int_0^L [\sigma(v^1(s,y)) - \sigma(v^2(s,y))] P_{t-s}(x,y)\, \xi(dy,ds)$$

$$F(t,x) = \mathbb{E}u^2(t,x) \qquad H(t) = \sup_{x\in[0,L]} F(t,x)$$

$$F(t,x) = \int_0^t \int_0^L \mathbb{E}\left[(\sigma(v^1(s,y)) - \sigma(v^2(s,y))\right]^2 P^2_{t-s}(x,y)\,dy\,ds$$

$$\leq \lambda^2 \int_0^t \int_0^L F(s,y) P^2_{t-s}(x,y)\,dy\,ds$$

$$\leq \lambda^2 C \int_0^t \frac{H(s)}{\sqrt{t-s}}\,ds \qquad \left[\int_0^L P^2_{t-s}(x,y)\,dy \leq \frac{C}{\sqrt{t-s}} \right]$$

$$\Rightarrow \quad H(t) \leq \lambda^2 C \int_0^t \frac{H(s)}{\sqrt{t-s}}\,ds \underset{\text{Iterating}}{\leq} (\lambda^2 C)^2 \int_0^t \frac{1}{\sqrt{t-s}} \left(\int_0^s \frac{H(u)}{\sqrt{s-u}}\,du \right) ds$$

$$\Leftrightarrow \quad = \lambda^4 C^2 \int_0^t du\, H(u) \int_u^t \frac{ds}{\sqrt{s-u}\sqrt{t-s}} \qquad \left[\text{Constants may depend on } T \right]$$

$$H(t) \leq 4\lambda^4 C^2 \int_0^t H(u)\,du \underset{\text{Gronwall's}}{\Longrightarrow} H \equiv 0$$

$\Rightarrow$ w.p. 1 $\quad u^2 \equiv 0 \quad \Rightarrow$ w.p. 1 $\quad v^1 \equiv v^2$ a.e.

INTERACTING DIFFUSIONS: $\quad i \in \mathbb{Z}, \quad dX^i_t = \Delta^D X^i_t + dB^i_t$

$$= (X^{i+1}_t + X^{i}_t - 2X^i_t) + dB^i_t$$

A sol^n exists as a Markov process in $B = \{f : \mathbb{Z} \to \mathbb{R} \mid \sum_{k\in\mathbb{Z}} 2^{-|k|} f(k) < \infty\}$ (Banach space)

[Shiga-Shimizu].

Does X^i_t on $\varepsilon\mathbb{Z}$ approximate $\dfrac{\partial u}{\partial t} = \Delta u + \xi$?

$\varepsilon\mathbb{Z} \ni i$

$i \in \mathbb{Z}$

$$dX_t^{\varepsilon,i} = \Delta^{(\varepsilon,D)} X_t^{\varepsilon,t}\,dt + \frac{1}{\sqrt{\varepsilon}}\,dB_t^{\varepsilon,i}$$

$$\Delta^{\varepsilon,D} X_t^{\varepsilon,i} = \frac{1}{\varepsilon^2}\left[X_t^{\varepsilon,i+1} + X_t^{\varepsilon,i-1} - 2X_t^{\varepsilon,i}\right]$$

"THEOREM".

$$\sup_{x\in[-M,M]}\ \sup_{t\in[0,T]}\left|X_t^{\varepsilon,\varepsilon[\frac{x}{\varepsilon}]} - u(t,x)\right| \longrightarrow 0 \quad \text{in Probability}$$

as $\varepsilon \to 0$.

[Initial conditions not discussed].

$i \in \mathbb{Z}$,

$$Y_i^{t,\varepsilon} = u(t,\varepsilon i) = \int_0^t\int_{\mathbb{R}} p_{t-s}(\varepsilon i - y)\,\xi(dy,ds)$$