1. Consider $\Gamma$ to be the join of two copies of $\mathbb{Z}^3$ at their origins. Write $\mathbb{Z}^3_{(i)}, i = 1, 2$ the two copies, and $0_i$ for their origins. Let
$$F = \{X \text{ is ultimately in } Z^3_{(1)}\}$$
and let $h(x) = \mathbb{P}^x(F)$.
   (a) Show that $h$ is harmonic,
   (b) Show that $h(x) \geq \mathbb{P}^x(X \text{ never hits } 0_1)$ for $x \in \mathbb{Z}^3_{(1)}$.
   (c) Show that $h(x) \leq \mathbb{P}^x(X \text{ hits } 0_2)$ for $x \in \mathbb{Z}^3_{(2)}$.
   (d) Conclude that $\Gamma$ does not satisty the Liouville Property

   Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Let $\Omega = V^{\mathbb{Z}_+}$. For any $n \geq 0$, let $X_n : \Omega \to V$ be given by $X_n(\omega) = \omega_n$,
$$\mathcal{F}_n = \sigma\{X_k : 0 \leq k \leq n\}, \mathcal{G}_n = \sigma\{X_k : 0 \leq k \geq n\}, \text{ and } \mathcal{F} = \mathcal{G}_0 = \sigma\{X_n : n \geq 0\}.$$

   **Random Walk on** $(\Gamma, \mu)$ **:** For any $x \in V$ let $\mathbb{P}^x$ be the unique measure on $(\Omega, \mathcal{F})$ such that
$$\mathbb{P}^x(X_0 = x_1, X_1 = x_2, \ldots, X_n = x_n) = 1_x(x_0)\prod_{i=1}^{n}\mathcal{P}(x_{i-1}, x_i),$$
   where $x_i \in V$ and $\mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x}$. For $x \in$, a $\sigma$-field $\mathcal{K}$ is $\mathbb{P}^x$ trivial if $\mathbb{P}^x(K)\{0, 1\}$ for all $K \in \mathcal{K}$.

2. Let $X_n$ be a random walk on $\mathbb{Z}$.
   (a) Show that $L = \sup\{n \geq 1 : X_n = 1\}$ is not a stopping time.
   (b) Show that $F = \inf\{n \geq 1 : X_n \in \{0, 4\}\}$ is a stopping time. Can you find the distribution of $X_F$ ?
   (c) Show that the return time $T_i$ to a state $i \in S$ is a stopping time.
   (d) Let $T_a = \inf\{n \geq 1 : X_n = a\}$. Show that $T_a$ is a stopping time and the 'inf' in $T_a$ is actually a minimum almost everywhere.
   (e) **(Reflection Principle)** Suppose $M_n = \max_{0 \leq i \leq n} X_i$. Show that
$$P(M_n \geq a, X_n < a) = P(M_n \geq a, X_n > a).$$
   (Hint: Apply the strong markov property at $T_a$ and symmetry of the distribution of the Bernoulli trials.)