Homework Set 3

1. Consider $\mathbb{Z}^d$ for $d \geq 1$ with natural weights and the random walk X_n on it. Let $A \subset \mathbb{Z}^d$ and define $\tau_A = \inf\{n \geq 0 : X_n \notin A\}$.
   (a) Let $d = 1$, $a < 0 < b$, $A = (a, b)$ Show that $\mathbb{P}^0(\tau_A > n(b-a)) \leq (1 - \frac{1}{2^{b-a}})^n$.
   (b) Let $\emptyset \neq A \subset \mathbb{Z}^d$ such that $\mid A \mid < \infty$. Then show that for any $x \in \mathbb{Z}^3$,
   $$\mathbb{P}^x(\tau_A > n) \leq c_1 \rho^n,$$
   for some $c_1 > 0$ and $0 < \rho < 1$.

2. Consider the graph $\Gamma = (V, E)$ formed by joining two copies of $\mathbb{Z}^3$ at the origin. We shall refer to $\mathbb{Z}^3_{(1)}$ and $\mathbb{Z}^3_{(2)}$ as the two copies. Show that Γ does not satisfy the Liouville Property.
   (Hint: With $F = \{\{X_n\}_{n \geq 0}$ is eventually in $\mathbb{Z}^3_{(1)}\}$, show $h : V \to [0.1]$ be given by $h(x) = \mathbb{P}^x(F)$ is harmonic on V and non-constant.)

   Check out recent Work on Liouville Property.

3. The invariant σ-field $\mathcal{I}$ is given by
   $$\mathcal{I} = \{F \in \mathcal{F} : \theta_n^{-1}(F) = F \text{ for all } n\}.$$
   The tail[1] σ-field $\mathcal{T}$ is given by
   $$\mathcal{T} = \cap_{n=1}^{\infty} \mathcal{G}_n.$$
   Show that $\mathcal{I} \subset \mathcal{T}$.
   (Hint: first show that $F \in \mathcal{G}_n$ iff $F = \theta_n^{-1}(F_n)$ for some $F_n \in \mathcal{G}_0$.)

4. Show that Γ satisfies the Liouville property if and only if $\exists x \in V$ such that $\mathcal{I}$ is $\mathbb{P}^x$ trivial.
   (Hint: For $\Longrightarrow$ for $F \in \mathcal{I}$ show that $h : V \to [0, 1]$ given by $h(x) = \mathbb{P}^x(F)$ is harmonic and $\Longleftarrow$ Use Martingale convergence Theorem.)

[1] Think of Example of : an event in $\mathcal{I}$; an event in $\mathcal{T}$; and an event not in $\mathcal{I}$.

Book-Keeping Results

Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Let $\Omega = V^{\mathbb{Z}_+}$. For any $n \geq 0$, let $X_n : \Omega \to V$ be given by $X_n(\omega) = \omega_n$,

$$\mathcal{F}_n = \sigma\{X_k : 0 \leq k \leq n\}, \mathcal{G}_n = \sigma\{X_k : 0 \leq k \geq n\}, \text{ and } \mathcal{F} = \mathcal{G}_0 = \sigma\{X_n : n \geq 0\}.$$

Random Walk on (Γ, μ) **:** For any $x \in V$ let $\mathbb{P}^x$ be the unique measure on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}^x(X_0 = x_1, X_1 = x_2, \ldots, X_n = x_n) = 1_x(x_0) \prod_{i=1}^{n} \mathcal{P}(x_{i-1}, x_i),$$

where $x_i \in V$ and $\mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x}$. For $x \in$, a σ-field $\mathcal{K}$ is $\mathbb{P}^x$ trivial if $\mathbb{P}^x(K)\{0, 1\}$ for all $K \in \mathcal{K}$. Let $I \subset \mathbb{Z}$. Let $X = \{X_n : n \in I\}$ be a stochastic process on a filtered probability space.

Martingale: We say X is a martingale if: (a) X is in L_1, so that $E[X_n] < \infty$ for each n, (b)X_n is $\mathcal{F}_n$ measurable for all n and (c)

$$E[X_n \mid \mathcal{F}_m] = X_m$$

for each $m \leq n$ and $m, n \in I$.

1. **(Optional Sampling Theorem)** Let $\{X_n : n \geq 0\}$ be a martingale and T be a stopping time. Suppose one of the following conditions holds:
   (a) T is bounded random variable,
   (b) X is bounded

   Then
   $$E(X_T) = E(X_0).$$

2. **(Martingale Convergence Theorem)** Let X be a martingale bounded in L_1.
   (a) If $I = \mathbb{Z}_+$ then there exists a random variable X_∞ with $\mathbb{P}(\mid X_\infty \mid < \infty) = 1$ such that $X_n \to X_\infty$ a.s. as $n \to \infty$.
   (b) If $I = \mathbb{Z}_-$ then there exists a random variable $X_{-\infty}$ with $\mathbb{P}(\mid X_{-\infty} \mid < \infty) = 1$ such that $X_n \to X_{-\infty}$ a.s. as $n \to -\infty$.

3. **(Uniform Integrability)** If $\{X_n\}$ is uniformly integrable if for all $\epsilon > 0$ there exists K such that
   $$E[\mid X_n \mid ; \mid X_n \mid > K] < \epsilon$$
   for all $n \geq 0$.
   (a) Bounded in L_1 does not imply that $\{X_n\}$ is uniformly integrable.
   (b) Let $\{X_n\}$ be uniformly integrable martingale. Then there exists X_∞ such that $X_n \to X_\infty$ a.s. and in L_1.

   The above can be found in the book:
   [D] Probability Theory and Examples, R. Durrett