Homework Set 2

1. Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Describe all the reversible measures for the random walk X_n on (Γ, μ).
2. Provide an example of a Markov Chain X_n on $(\Gamma = (V, E), \mu)$ where μ is a measure on V such that μ is stationary but not reversible.
3. Let $\mid V \mid < \infty$. Show that all harmonic functions are constants.
4. Let $\Gamma = (V, E)$ where $V = \{0, 1, \ldots, n\}$ and $E = \{\{i, i+1\} : 0 \leq i \leq n-1\}$. Let μ be the natural weights on Γ.
   (a) Suppose h is a harmonic function on $V \setminus \{0, n\}$, such that $h(0) = 1$ and $h(n) = 0$. Find h.
   (b) T_a be the hitting time of the point a. Suppose $h : V \to [0, 1]$ is given by $h(i) = \mathbb{P}^i(T_n < T_0)$. Calculate h explicitly.
   (c) Suppose $g : V \to [0, \infty)$ be given by $g(i) = \mathbb{E}^i[\min(T_0, T_n)]$. Calculate g explicitly.
5. Let (Γ, μ) be a recurrent graph. Let $\rho \in V, n \geq 1$, and $B(\rho, n)$ be the ball of radiuse n around ρ. Let $\tau_{B(\rho,n)} = \min\{k \geq 0 : X_k \notin B(\rho, n)\}$. Define $h_n : V \to [0, 1]$ by
$$h_n(x) = \mathbb{P}^x(\tau_{B(\rho,n)} < T_\rho).$$
   Show that h_n is harmonic on $B(\rho, n) \setminus \{\rho\}$and $\lim_{n\to\infty} h_n(x) = 0$ for all $x \in V$.
6. Let $V = \mathbb{Z}^3$ be equipped with the cannonical edges and natural weights. Let X_n be the random walk on it.
   (a) Show that X_n is transient on $\mathbb{Z}^3$.
   (b) Let $n \geq 1$, $A = \mathbb{Z}^3 \setminus \{0\}$. Let $h_n, h : V \to [0, 1]$ be given by
$$h_n(x) = \mathbb{P}^x(T_0 \geq n) = \mathbb{P}^x(X_k \in A, 1 \leq k \leq n)$$
and
$$h(x) = \mathbb{P}^x(T_0 = \infty) = \mathbb{P}^x(X_n \in A, \text{ for all } n \geq 0).$$
      i. Show that $h_n = Q^n 1_A$ and $h = Qh$, where Q is the restriction of P onto A.
      ii. Suppose $\alpha = \sup_{x\in A} h(x)$, show that $0 < \alpha \leq 1$ and $h \leq \alpha 1_A$
      iii. Using (i) and (ii), conclude that $h \leq \alpha h_n$
      iv. Conclude that $\max_{x\in\partial A} h(x) \neq \sup_{x\in\bar{A}} h(x)$.

Book-Keeping Exercises

Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Let $\Omega = V^{\mathbb{Z}_+}$. For any $n \geq 0$, let $X_n : \Omega \to V$ be given by $X_n(\omega) = \omega_n$, $\mathcal{F}_n = \sigma\{X_k : 0 \leq k \leq n\}$ and $\mathcal{F} = \sigma\{X_n : n \geq 0\}$. For any $x \in V$ let $\mathbb{P}^x$ be the unique measure on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}^x(X_0 = x_1, X_1 = x_2, \ldots, X_n = x_n) = 1_x(x_0) \prod_{i=1}^{n} \mathcal{P}(x_{i-1}, x_i),$$

where $x_i \in V$ and $\mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x}$.

1. Show that the measure μ on V is stationary for X_n, that is

$$\mu(\{y\}) = \sum_{x \in V} \mu(\{x\})\mathcal{P}(x, y).$$

2. Show that the measure μ is said to be reversible if

$$\mu(\{y\})\mathcal{P}(x, y) = \mu(\{x\})\mathcal{P}(x, y), \text{ for all } x, y \in V.$$

3. Let $g : V \times V \to [0, \infty) \cup \{\infty\}$ be the Green function for X_n on (Γ, μ), Show that the following are equivalent

   (a) (Γ, μ) is recurrent.

   (b) $g(x, y)$ is infinite for all $x, y \in V$.

   (c) $\exists\, x, y$ such that $g(x, y) = \infty$.

4. Suppose $x \subset V$. Let $T_x^1 = T_x^+$ be the return time to x. Let

$$T_x^k = \min\{n > T_x^{k-1} : X_n \in A\}, \text{ for } k \geq 2$$

   be the successive return times to x. Let $y \neq x$ and $y \in V$, Define

$$Y_k = I(T_y < T_x) \circ \theta_{T_x^k}.$$

   Find the distribution of Y_k.