Homework Set 1

1. Let (X_i, d_i), $i = 1, 2$ be metric spaces. A map $\phi : X_1 \to X_2$ is a rough isometry if there exist positive constants C_1, C_2 such that

$$\frac{1}{C_1}(d_1(x,y) - C_2) \leq d_2(\phi(x), \phi(y)) \leq C_1(d_1(x,y) + C_2)$$

and

$$\cup_{x \in X_1}\{y \in X_2 : d_2(\phi(x), y) \leq C_2\} = X_2.$$

If there exists a rough isometry between two spaces they are said to be roughly isometric.

(a) Show that $\mathbb{Z}^d, \mathbb{R}^d, [0,1] \times \mathbb{R}^d$ are roughly isometric spaces.

(b) Let G be a finitely generated infinite group, and λ, Λ' be two sets of generators. Let Γ, Γ' be the associated Cayley graphs. Then Γ, Γ' are roughly isometric.

2. Let $(\Gamma_i, \mu_i), i = 1, 2$ be weighted graphs and have controlled weights, i.e.

$$\frac{\mu_{xy}}{\mu_x} \geq \frac{1}{C_2} \text{ whenever } x \sim y.$$

A map $\phi : V_1 \to V_2$ is a rough isometry between (Γ_1, μ_1) and (Γ_1, μ_2) if:

(i) ϕ is a rough isometry between the metric spaces (V_1, d_{Γ_1}) and (V_2, d_{Γ_2}) (with constants C_1 and C_2).

(ii) There exists $0 < C_3 < \infty$ such that for all $x \in V_1$

$$\frac{1}{C_3}\mu_1(x) \leq \mu_2(\phi(x)) \leq C_3\mu_1(x)$$

(a) Let $\alpha > 0$. Consider the graph $\mathbb{Z}_+$ with weights $\mu^{\alpha}_{n,n+1} = \alpha^n$.

   i. Show the graph $(\mathbb{Z}_+, \mu^{\alpha})$ has controlled weights. Does it have bounded weights ?
   ii. Show that the graph is recurrent if and only if $\alpha \leq 1$.
   iii. For $\alpha \neq \beta$ are $(\mathbb{Z}_+, \mu^{\alpha})$ and $(\mathbb{Z}_+, \mu^{\beta})$ roughly isometric ?

3. Let $(\Gamma = (V, E), \mu)$ be transient. Let $f : V \to \mathbb{R}$ be given by $f(y) = 1$ for all $y \in V$. Let $x \in V$ and $g^x(\cdot)$ be the Green function on V.

(a) Show that $f \in H^2$ and $g \in H^2_0$

(b) Verify that $\mathcal{E}(f, g^x)$ is well defined and compute it.

(c) Find Δg^x and Δf.

(d) Find $\langle \Delta f, g^x \rangle$ and $\langle f, \Delta g^x \rangle$.

(e) Comment on whether discrete Gauss-Green Theorem holds or not for f, g^x.

Book-Keeping Exercises

Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Let $\Omega = V^{\mathbb{Z}_+}$. For any $n \geq 0$, let $X_n : \Omega \to V$ be given by $X_n(\omega) = \omega_n$, $\mathcal{F}_n = \sigma\{X_k : 0 \leq k \leq n\}$ and $\mathcal{F} = \sigma\{X_n : n \geq 0\}$. For any $x \in V$ let $\mathbb{P}^x$ be the unique measure on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}^x(X_0 = x_1, X_1 = x_2, \ldots, X_n = x_n) = 1_x(x_0) \prod_{i=1}^{n} \mathcal{P}(x_{i-1}, x_i),$$

where $x_i \in V$ and $\mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x}$.

1. Let ξ be a bounded function and measurable with respect to $\mathcal{F}_n$. Let η be a bounded function and measurable with respect to $\mathcal{F}$. For $n \geq 0$, let $\theta_n : \Omega \to \Omega$ be given by

   $$\theta_n(\omega) = (\omega_n, \omega_{n+1}, \ldots).$$

   Show that Markov property holds for X_n i.e.

   $$\mathbb{E}^x[\xi(\eta \circ \theta_n) \mid \mathcal{F}_n] = \mathbb{E}^x[\xi g(X_n)],$$

   where $g(y) = \mathbb{E}^y[\eta]$.

2. For $z \in V$, let $T_z = \min\{n \geq 0 : X_n = z\}$ and $T_z^+ = \min\{n \geq 0 : X_n = z\}$. Show that the following conditions are equivalent:
   - (T1) $\exists x \in V$ such that $\mathbb{P}^x(T_x^+ < \infty) < 1$.
   - (T2) For all $x \in V$, $\mathbb{P}^x(T_x^+ < \infty) < 1$.
   - (T3) For all $x \in V$, $\sum_{n=0}^{\infty} \mathbb{P}^x(X_n = x) < \infty$.
   - (T4) For all $x, y \in V$ with $x \neq y$, $\mathbb{P}^x(T_y < \infty) < 1$ or $\mathbb{P}^x(T_y < \infty) < 1$.
   - (T5) For all $x, y \in V$ with $x \neq y$, $\mathbb{P}^x(\sum_{n=0}^{\infty} 1(X_n = y) < \infty) = 1$

3. Show that $C_o(V) = \{f : V \to \mathbb{R} : \text{Supp}(f) \text{ is finite }\}$ is dense in $L^p(V, \mu)$ for all $p \in [1, \infty]$.

4. For all $p, r \in [1, \infty]$, let $A : L^p(V, \mu) \to L^r(V, \mu)$. Show that

   $$\| A \|_{p \to q} = \sup\{\| Af \|_q : \| f \|_p \leq 1\}$$

   is a norm.

5. Let $C(V) = \{f : V \to \mathbb{R}\}$. Show that $P_n : C(V) \to C(V)$ defined by

   $$P_n f(x) = \sum_{n=0}^{\infty} \mathbb{P}^x(X_n = y) f(y),$$

   satisfies $P_n = (P_1)^n$

6. Let $H^2(V) = \{f \in C(V) : \mathcal{E}(f, f) < \infty\}$ where

   $$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu_{xy}(f(x) - f(y))(g(x) - g(y)) \text{ and } \| f \|_{H^2} = \mathcal{E}(f, f) + f(\rho)^2,$$

   for $f, g \in C(V)$ and (fixed) $\rho \in V$.

   (a) Show that $H^2(V)$ is a Hilbert space.

   (b) Let $f_n \in H^2$ with $\sup_n \| f_n \|_{H^2} < \infty$. Then there exists $\{f_{n_k}\}$ and $f \in H^2$ such that for each $x \in V$,

   $$\lim_{k \to \infty} f_{n_k}(x) = f(x) \text{ and } \| f \|_{H^2} \leq \liminf_{k \to \infty} \| f_{n_k} \|_{H^2}$$