Due: Thursday, April 10th, 2014
Problem to be turned in: 4

Modes of Convergence:

- **Convergence in Probability:** A sequence of random variables X_n converges to a random variable X in Probability if for any $\epsilon > 0$,
$$\lim_{n\to\infty} P(\mid X_n - X \mid > \epsilon) = 0.$$
This is denoted by $X_n \stackrel{p}{\to} X$.
- **Almost everywhere convergence:** A sequence of random variables X_n converges to a random variable X almost everywhere if,
$$P(\lim_{n\to\infty} X_n = X) = 1.$$
This is denoted by $X_n \stackrel{a.e.}{\longrightarrow} X$.
- **Convergence in Distribution:** A sequence of random variables X_n (with distribution functions F_n) converges to a random variable X (with distribution function F) in distribution if
$$\lim_{n\to\infty} F_n(x) = F(x),$$
whenever x is a continuity point of F. This is denoted by $X_n \stackrel{d}{\longrightarrow} X$.

1. Let $a_n = \sum_{k=0}^{n} \frac{n^k}{k!} e^{-n}$, $n \geq 1$. Using the Central Limit Theorem evaluate $\lim_{n\to\infty} a_n$.
2. Let $Y \stackrel{d}{=} N(0,1)$. Let $X_n = (-1)^n Y$. Discuss convergence a.e, in probability, and in distribution of X_n.
3. For $n \geq 1$, let $0 \leq p_n \leq 1$ and $\lim_{n\to\infty} p_n = 0$. Consider
$$X_n = \begin{cases} 1 & \text{w.p.} \quad p_n \\ 0 & \text{w.p.} \quad 1 - p_n. \end{cases}$$
Let $Y_n = \prod_{k=1}^{n} X_k$. Workout explicit conditions on the sequence $\{p_n\}$ that ensure
   (a) $Y_n \stackrel{p}{\to} 0$, or
   (b) $Y_n \stackrel{p}{\to} 1$, or
   (c) for any $0 \leq \alpha \leq 1, Y_n \stackrel{d}{\longrightarrow} Y$, where
$$Y = \begin{cases} 1 & \text{w.p.} \quad \alpha \\ 0 & \text{w.p.} \quad 1 - \alpha. \end{cases}$$
4. Let X_n have the t-distribution with n degrees of freedom. Show that $X_n \stackrel{d}{\longrightarrow} X$ where X is standard Normal distribution.