Due: Thursday, April 3rd, 2014

Problem to be turned in: 4,8

1. For $n \geq 1$, consider the function $E_n : \mathbb{C} \to \mathbb{C}$ given by
$$E_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$
Show that
   (a) $E_n(z)$ converges for all $z \in \mathbb{C}$ and Let $E(z) := \lim_{n\to\infty} E_n(z)$,
   (b) $E(z)E(w) = E(z+w)$ for $z, w \in \mathbb{C}$,
   (c) E is differentiable and $E'(z) = E(z)$ for all $z \in \mathbb{C}$.

2. Let E be defined above in (1). If $E(1) = e$, then show that for any real number x, $E(x) = e^x$.
   (a) Show that $\lim_{x\to\infty} x^n e^{-\alpha x} = 0$ for $n \in \mathbb{N}$ and $\alpha > 0$.
   (b) Show that E is a strictly positive increasing function on R. Denote its inverse by $L : (0,\infty) \to \mathbb{R}$.
   (c) Show that $L(y) = \int_1^y \frac{1}{x} dx$.

3. Let E be defined above in (1). Define for $x \in \mathbb{R}$, $C, S : \mathbb{R} \to \mathbb{R}$
$$C(x) = \frac{1}{2}[E(ix) + E(-ix)], \ S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$
   (a) Show that C and S are differentiable and $C' = -S$, $S' = C$.
   (b) Show that there exists $x > 0$ such that $C(x) = 0$.
   (c) Let $\frac{\pi}{2} = \min\{x > 0 : C(x) = 0\}$. Show that $S(\frac{\pi}{2}) = 1$.
   (d) Conclude that E is periodic with period $2\pi i$, S is periodic with period 2π and S is periodic with period 2π.
   (e) $z \in \mathbb{C}$ with $\mid z \mid= 1$ then there is a $t \in [0, 2\pi)$ such that $E(it) = z$.

4. The length of time (in appropriate units) that a certain type of component functions before failing is a random variable with probability density function
$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
Once the component fails it is immediately replaced with another one of the same type.
   (a) If we let X_i denote the lifetime of the i^{th} component to be put in use, then $S_n = \sum_{i=1}^{n} X_i$ represents the time of the n^{th} failure. The long–term rate at which failures occur is
$$r = \lim_{n\to\infty} \frac{n}{S_n}$$
   Determine r, assuming that the random variables X_i are independent.
   (b) How many components would one need to have on hand to be approximately 90% certain that the stock would last at least 35 units of time?

5. Two types of coin are produced at a factory: a fair coin and a biased one that comes up heads 55% of the time. We have one of these coins but do not know whether it is a fair or biased coin. In order to ascertain which type of coin we have, we shall perform the following statistical test. We shall toss the coin 1000 times. If the coin comes up heads 525 or more times we shall conclude that it is a biased coin. Otherwise, we shall conclude that it is fair. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

6. A small college has 2014 students. What is the probability that more than five students were born on independence day ? Assume that birthrates are constant throughout the year and that each year has 365 days.

7. A machine in a heavy–equipment factory produces steel rods of length Y, where Y is a normal random variable with a mean μ of 6 inches and a variance σ^2 of 0.2. The cost C of repairing a rod that is not exactly 6 inches in length is proportional to the square of the error and is given (in Rupees) by

$$C = 4(Y - \mu)^2$$

If 50 rods with independent lengths are produced in a given day, approximate the probability that the total cost for repairs for that day will exceed Rs 48.

8. Let $Y_1,\ Y_2,\ \cdots,\ Y_n$ be independent random variables, each uniformly distributed over the interval $(0, \theta)$.

   (a) Show that the mean $\bar{Y}$ converges in probability towards a constant as $n \to \infty$ and find the constant.

   (b) Show that $\max\{Y_1, \cdots, Y_n\}$ converges in probability toward θ as $n \to \infty$.