Due: Thursday, March 20th, 2014
Problem to be turned in: 2, 3(c)

1. Let $n \geq 1$ and $\{X_i\}_{i=1}^n$ be i.i.d. Normal random variables with mean μ and variance σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $s_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)$.
   (a) Construct an orthogonal matrix $T_{n\times n}$, (i.e. $TT^t = I = T^tT$) such that every element of first row of T is $\sqrt{\frac{1}{n}}$
   (b) Let $X = \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{bmatrix}$ and $Y = TX$. Find the distribution of X and Y.
   (c) Show that $Y_1 = \sqrt{n}\bar{X}_n$ and $\sum_{i=2}^n Y_i^2 = (n-1)s_n^2$.
   (d) Conclude that: $\bar{X}_n$ and s_n^2 are independent; and $\frac{n-1}{\sigma^2}s_n^2$ has χ^2_{n-1} distribution.
2. Let X be distrbuted as an Exponential (λ) random variable.
   (a) Find the moment generating function of X (wherever it exists).
   (b) Using the above find $E(Y^m)$ where $Y \stackrel{d}{=} \Gamma(n, \lambda)$ random variable, $m \geq 1$ and $n \geq 1$.
3. Suppose X and Y are two random variables having a joint density by

$$f(x,y) = \begin{cases} \frac{2e^{-\left(y+\frac{x}{y}\right)}}{y} & 0 \leq x < \infty, 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find p.d.f of Y
   (b) Find the conditional density of X given $Y = y$.
   (c) Find the conditional expectation of X given $Y = y$.
   (d) Find $\text{Cov}(X, Y)$
4. Suppose X has a probability density function given by

$$f_X(x) = \begin{cases} 5x^{-6} & 1 \leq x < \infty, \\ 0 & \text{otherwise} \end{cases}$$

   Find $k = \max\{n \in \mathbb{N} : E(X^n) < \infty\}$.