Due: Thursday, February 13th, 2014
Problem to be turned in: 1(d), 2(a), 3(a)

1. Let X, Y, and Z be absolutely continuous random variables, and let $a, b \in \mathbb{R}$. Then,
   (a) $Cov[X, Y] = Cov[Y, X]$;
   (b) $Var[X + Y] = Var[X] + Var[Y] + 2Cov[X, Y]$.
   (c) $Cov[X, aY + bZ] = a \cdot Cov[X, Y] + b \cdot Cov[X, Z]$
   (d) $Cov[aX + bY, Z] = a \cdot Cov[X, Z] + b \cdot Cov[Y, Z]$
   (e) If X and Y are independent with a finite covariance, then $Cov[X, Y] = 0$.
   (f) Let ρ be the correlation coefficient of X, Y. Show that $\rho^2 \in \{+1, -1\}$ if and only if there are $a, b \in \mathbb{R}$ with $a \neq 0$ for which $P(Y = aX + b) = 1$.
2. Using Moment generating functions :
   (a) Let $Y \sim$ Exponential(λ), calculate $E[Y^3]$ and $E[Y^4]$, the third and fourth moments of an exponential distriubtion.
   (b) For $i = 1, 2$ let $X_i \stackrel{d}{=}$ Normal(μ_i, σ_i^2) with X_1, X_2 independent. Let a_1, a_2 be real numbers, not all zero, and let $Y = a_1X_1 + a_2X_2$. Prove that Y is normally distributed and find its mean and variance in terms of the a's, μ's, and σ's.
   (c) Suppose X, Y are two random variables then distributions of all linear combinations of X, Y completely characterise the joint distribution of X and Y.
3. Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a Bivariate Normal random variable with mean vector μ and non-singular covariance matrix Λ.
   (a) Suppose $A_{2\times2}$ and $b_{2\times1}$ are real matrices. Let $Y = AX + b$. Let its mean vector be η and covariance matrix Σ. Show that $\eta = A\mu + b$, $\Sigma = A\Lambda A^T$
   (b) Show that X_1 and X_2 are independent if and only if $Cov(X_1, X_2) = 0$.
   (c) If X_1 and X_2 are independent then find the distribution of $W = \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}$
   (d) Let X_1 and X_2 have standard Normal distribution and correlation ρ. Find the distribution of Z with $Z = \frac{1}{1-\rho^2}(X_1^2 - 2\rho X_1X_2 + X_2^2)$.