Due: Thursday, January 23rd, 2014
Problem to be turned in: None.

1. A biologist is modeling the size of a frog population in a series of ponds. She is concerned with both the number of egg masses laid by the frogs during breeding season and the annual precipitation into the ponds. She knows that in a given year there is an 86% chance that there will be over 150 egg masses deposited by the frogs (event E) and that there is a 64% chance that the annual precipitation will be over 17 inches (event F).

   (a) In terms of E and F, what is the event "there will be over 150 egg masses and an annual precipitation of over 17 inches"?

   (b) In terms of E and F, what is the event "there will be 150 or fewer egg masses and the annual precipitation will be over 17 inches"?

   (c) Suppose the probability of the event from (a) is 59%. What is the probability of the event from 2b?

2. A box contains M balls, of which W are white. A sample of n balls, with $n \leq W$ and $n \leq M - W$, is drawn at random and **without** replacement. Let A_j, where $j = 1, 2, \cdots, n$, denote the event that the ball drawn on the j^{th} draw is white. Find $P(A_1)$, $P(A_2)$ and $P(A_3)$. Guess what $P(A_j)$ is.

3. In a test called Narco-Analysis, a "truth" serum is given to a suspect. It is known that it is 90% reliable when the person is guilty and 99% reliable when the person is innocent. In other words 10% of the guilty are judged innocent by the serum and 1% of the innocent are judged guilty. If the suspect was selected from a group of suspects of which only 5% have ever committed a crime and the serum indicates that she is guilty, what is the probability that she is innocent ?

4. *Polya Urn scheme*– An urn contains b black balls and r red balls. A ball is drawn at random. The ball is replaced into the urn along with c balls of its colour and d balls of the opposite colour. Then another random ball is drawn and the procedure is repeated.

   (a) What is the probability that the second ball drawn is a black ball ?
   (b) Assume $c = d$. What is the probability that the second ball drawn is a black ball ?
   (c) Assume $c = d$. What is the probability that the n^{th} ball drawn is a black ball ?

5. A box contains M balls, of which W are white. A sample of n balls is drawn at random, **with** replacement. Let A_j, where $j = 1, 2, \cdots, n$, denote the event that the ball drawn on the j^{th} draw is white. Let B_k denote the event that the sample of n balls contains exactly k white balls. Find $P(A_j | B_k)$.

6. Suppose that airplane engines operate independently in flight and fail with probability p $(0 \leq p \leq 1)$. A plane makes a safe flight if at least half of its engines are running. Kingfisher Air lines has a four–engine plane and Paramount Airlines has a two–engine plane for a flight from Bangalore to Delhi. Which airline has the higher probability for a successful flight ?

7. At a Doordarshan National TV opinion poll, they wish to know the percentage p of people who intend to vote for congress. How large must a random sample with replacement be in order to be at least 95% sure that the sample percentage is within one percent of p ?

8. Suppose that the number of earthquakes that occur in a year, anywhere in the world, is a Poisson random variable with mean λ. Suppose that the probability that any given earthquake has magnitude at least 5 on the Richter scale is p. Find the probability that there are n earthquakes with magnitude at least 5 in a year.

9. Suppose Ω is a sample space consisting of sequences of two coin flips. Let X be a r.v. that is 1 if the first coin is heads, and 0 otherwise, Y be a r.v. that is 1 if the first coin is tails, and 0 otherwise, and Z be a r.v. that is 1 if the second coin is tails, and 0 otherwise. (a) Show that X, Y, and Z all have the same p.m.f. (b) Show that the pairs (X, Y) and (X, Z) have different joint p.m.f.s. (c) Are X and Y independent? Why or why not? (d) Are X and Z independent? Why or why not?

10. Suppose tickets numbered $\{1, 2, \ldots, n\}$ are placed in a box and drawn one by one at random without replacement. Let X_i be the number of the ith ticket drawn, $1 \leq i \leq n$. (a) Find the joint distribution of $(X_1, X_2, \ldots, X_n\}$. (b) Find the distribution X_j for $1 \leq j \leq n$.

11. Let X and Y be independent random variables each geometrically distributed with parameter p. (a) Find the distribution of $\min(X, Y)$. (b) Find $P(\min(X, Y) = X)$. (b) Find the distribution of $X + Y$. (c) Find the $P(X > m + n | X > n)$. (d) Find $P(Y = y | X + Y = z)$.

12. A box contains red and black balls. First, 10 balls are drawn with replacement and let X_1 be the number of red balls. Next, 10 balls are drawn with replacement and let X_2 be the number of red balls in this second sample. Are X_1 and X_2 independent ?

13. Suppose we have a population of d elements. Let $n \leq d$. We draw a sample with replacement until exactly n distinct elements have been obtained. Let X_n denote the size of the sample required. Find $E(X_n)$.

14. Let X be a discrete random variable with $E(X) = 10$. What is the largest possible value of $P(X \geq 1000)$?

15. Let X and Y be two discrete random variables. When are $X+Y$ and $X-Y$ are uncorrelated ?

16. Let Y be a point chosen random from $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $X = \tan(Y)$. Find the probability density function of X.

17. Let G_n be a discrete random variable with distribution Geometric$(\frac{1}{n})$. Let $t \in \mathbb{R}$. Find the $\lim_{n\to\infty} P(G_n > \frac{t}{n})$.

18. Let X be a random variable with distribution function F. Let $Y = F(X)$. Find the distribution of Y.

19. Let X be an Exponential (λ) random variable. For $t_0 > 0$, define $Y = X 1_{(X \leq t_0)} + t_0 1_{(X > t_0)}$. Find the distribution function of Y. Convince yourself that Y is neither discrete nor an absolutely continuous random variable.

20. Let T be an exponential random variable with mean λ. Find the probability density function of $Y = \frac{T}{1+T}$.