Ground Rules: Time allowed is 20 minutes, individual work only and closed book test.

Your name ______________________ **Score :**

Let $X \sim$ Uniform $\{1,2,3\}$ be independent of $Y \sim$ Uniform $\{1,2,3\}$. Let $Z = \max(X,Y)$ and $W = \min(X,Y)$. Find $E[Z \mid W]$.

$E(Z/W=w)$, $w \in \{1,2,3\}$.

$$= \sum_{k=1}^{3} k\mathbb{P}(Z=k | W=w)$$

$$= \sum_{k=1}^{3} k \frac{\mathbb{P}(Z=k, W=w)}{\mathbb{P}(W=w)}$$

$$= \sum_{k=w}^{3} k \frac{\mathbb{P}(Z=k; W=w)}{\mathbb{P}(W=w)}$$

$$= \sum_{k=w}^{3} k \frac{\mathbb{P}(X=k, Y=w) + \mathbb{P}(Y=k, X=w)}{\mathbb{P}(W=w)}.$$

We find joint distribution of X & Y

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\ \mathbb{P}(Y=y) = \frac{1}{9}$$

(independence) $x \in \{1,4,3\}$, $y \in \{1,4,3\}$

	$Y=1$	$Y=2$	$Y=3$
$X=1$	$1/9$	$1/9$	$1/9$
$X=2$	$1/9$	$1/9$	$1/9$
$X=3$	$1/9$	$1/9$	$1/9$

$$\mathbb{P}(W=1) = \mathbb{P}(X=1, Y\geq 1) + \mathbb{P}(Y=1, X \gneq 1)$$

$$= \sum_{k=1}^{3} \mathbb{P}(X=1, Y=k) + \sum_{k=2}^{3} \mathbb{P}(Y=1, X=k)$$

$$= 3\cdot\left(\tfrac{1}{9}\right) + 2\left(\tfrac{1}{9}\right) = \frac{5}{9}$$

$$\mathbb{P}(W=2) = \mathbb{P}(X=2, Y\geq 2) + \mathbb{P}(Y=2, X \gneq 2)$$

$$= 2\,\tfrac{1}{9} + 1\cdot\tfrac{1}{9}$$

$$= 3/9$$

$$\mathbb{P}(W=3) = \mathbb{P}(X=3, Y=3) = 1/9$$

$$\therefore\ E(Z \mid W=w) = \sum_{k=w}^{3} k\ \tfrac{2}{9}\cdot\frac{1}{\mathbb{P}(W=w)}$$

$\boxed{W=1}$ $$E(Z \mid W=1) = \sum_{k=1}^{3} k\ \frac{2}{9}\cdot\frac{9}{5}$$

$\boxed{W=2}$ $E(Z|W=2) = \sum_{k=2}^{3} k \cdot \frac{2}{9} \cdot \frac{9}{3}$

$= \frac{2}{3} \cdot 5 = \frac{10}{3}$

$\boxed{W=3}$ $E(Z|W=3) = 3 \cdot \frac{2}{9} \cdot 9 = 6$

$\cancel{E(Z|W) = \sum_{w=1}^{3} w \; E(Z|W=w)}$

$\cancel{= 1 \cdot \frac{12}{5} + 2 \cdot \frac{10}{3} + 3 \cdot 6}$

$\cancel{= \frac{36 + 100 + 270}{15}}$

$\square$

$E[Z|W] = \sum_{w=1}^{3} g(w) \, \mathbb{P}(W=w)$

where $g(w) = E(Z|W=w)$.

$= \sum_{w=1}^{3} g(1) \, \mathbb{P}(W=1) + g(2) \, \mathbb{P}(W=2)$

$+ g(3) \, \mathbb{P}(W=3)$

$= \frac{12}{5} \cdot \frac{5}{9} + \frac{10}{3} \cdot \frac{3}{9} + \frac{6 \cdot 1}{9} = \frac{28}{9}$