Due: Thursday, October 6th 2016
Problem to be turned in: 3,8

1. A standard light bulb has an average lifetime of four years with a standard deviation of one year. A Super D-Lux lightbulb has an average lifetime of eight years with a standard devaition of three years. A box contains many bulbs – 90% of which are standard bulbs and 10% of which are Super D-Lux bulbs. A bulb is selected at random from the box. What are the average and standard deviation of the lifetime of the selected bulb?

2. Let X and Y be described by the joint distribution

	$X = -1$	$X = 0$	$X = 1$
$Y = -1$	1/15	2/15	2/15
$Y = 0$	2/15	1/15	2/15
$Y = 1$	2/15	2/15	1/15

and answer the following questions.

(a) Calculate $E[X|Y = -1]$.

(b) Calculate $Var[X|Y = -1]$.

(c) Describe the distribution of $E[X|Y]$.

(d) Describe the distribution of $Var[X|Y]$.

3. Let X and Y be discrete random variables. Let x be in the range of X and let y be in the range of Y.

(a) Suppose X and Y are independent. Show that $E[X|Y = y] = E[X]$ (and so $E[X|Y] = E[X]$).

(b) Show that $E[X|X = x] = x$ (and so $E[X|X] = X$).

4. Let $X \sim$ Uniform $\{1, 2, \ldots, n\}$ be independent of $Y \sim$ Uniform $\{1, 2, \ldots, n\}$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.

(a) Find the joint distribution of (Z, W).

(b) Find $E[Z \mid W]$.

5. Consider the experiment of flipping two coins. Let X be the number of heads among the coins and let Y be the number of tails among the coins.

(a) Should you expect X and Y to be posivitely correlated, negatively correlated, or uncorrelated? Why?

(b) Calculate $Cov[X, Y]$ to confirm your answer to (a).

6. Let X, Y, and Z be discrete random variables, and let $a, b \in \mathbb{R}$. Then,

(a) $Cov[X, Y] = Cov[Y, X]$;

(b) $Cov[X, aY + bZ] = a \cdot Cov[X, Y] + b \cdot Cov[X, Z]$;

(c) $Cov[aX + bY, Z] = a \cdot Cov[X, Z] + b \cdot Cov[Y, Z]$; and

7. Let X and Y be two discrete random variables both with finite variances σ_x and σ_y. Let their means be μ_x and μ_y respectively. Then

(a) Using $0 \leq E[(\frac{X-\mu_X}{\sigma_X} - \frac{Y-\mu_Y}{\sigma_Y})^2]$ show that $Cov[X, Y] \leq \sigma_X \sigma_Y$.

(b) Show that $Cov[X,Y] \geq -\sigma_X\sigma_Y$ and xonclude that the correlation cooefficient $\rho[X,Y] = \frac{Cov[X,Y]}{\sigma_X\sigma_Y}$ is such that

$$-1 \leq \rho[X,Y] \leq 1$$

.

(c) Prove that $\rho[X,Y] = \pm 1$ if and only if there are $a, b \in \mathbb{R}$ with $a \neq 0$ for which $P(Y = aX + b) = 1$.

8. Let $X \sim \text{Uniform}(\{0,1,2\})$ and let Y be the number of heads in X flips of a coin.

(a) Should you expect X and Y to be positively correlated, negatively correlated, or uncorrelated? Why?

(b) Calculate $Cov[X,Y]$ to confirm your answer to (a).

9. Let X and Y be discrete random variables with finite variances.

   (a) Show that
   $$Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y].$$

   (b) Use (a) to conclude that when X and Y are positively correlated, then $Var[X+Y] > Var[X] + Var[Y]$, while when X and Y are negatively correlated, $Var[X+Y] < Var[X] + Var[Y]$.

   (c) Suppose X_i $1 \leq i \leq n$ are discrete random variables with finite variance and covariances. Use induction and (a) to conclude that

$$Var[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} Var[X_i] + 2 \sum_{1 \leq i < j \leq n} Cov[X_i, X_j].$$