Due: Thursday, October 20th, 2016
Problem to be turned in: 2, 3

1. Let X be a random variable with density $f(x) = 3x^2$ for $0 < x < 1$ (and $f(x) = 0$ otherwise). Calculate the distribution function of X.

2. Let $X \sim \text{Uniform}(0, 1)$.
   (a) Let $Y = \sqrt{X}$. Determine the density of Y.
   (b) Let $Z = \frac{1}{X}$. Determine the density of Z.
   (c) Let $r > 0$ and define $Y = rX$. Show that Y is uniformly distributed on $(0, r)$.
   (d) Let $Y = 1 - X$. Show that $Y \sim \text{Uniform}(0, 1)$ as well.
   (e) Let a and b be real numbers with $a < b$ and let $Y = (b-a)X+a$. Show that $Y \sim \text{Uniform}(a, b)$.
   (f) Find a function $g(x)$ (which is strictly increasing) such that the random variable $Y = g(X)$ has density $f_Y(y) = 3y^2$ for $0 < y < 1$ (and $f_Y(y) = 0$ otherwise).

3. If $X \sim \text{Normal}\ (0, 1)$. Let $Y = X^2$. Find the density function of Y

4. Let $\alpha > 0$ and X be a random variable with the p.d.f given by

$$f(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & 1 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

   The random variable X is said to have Pareto (α) distribution.
   (a) Find the distribution of $X_1 = X^2$
   (b) Find the distribution of $X_2 = \frac{1}{X}$
   (c) Find the distribution of $X_3 = \ln(X)$

   In the above exercises we assume that the transformation function is defined as above when the p.d.f of X is positive and zero otherwise.

5. Let X be a continuous random variable with probability density function $f_X :\to \mathbb{R}$. Let $a > 0$, $b \in \mathbb{R}$ $Y = \frac{1}{a}(X - b)^2$. Show that Y is also a continuous random variable with probability density function $f_Y : \mathbb{R} \to \mathbb{R}$ given by

$$f_Y(y) = \frac{1}{2\sqrt{ay}}\left[f_X(\sqrt{ay} + b) - f_X(-\sqrt{ay} + b)\right]$$

6. Let $-\infty \leq a < b \leq \infty$ and $I = (a, b)$ and $g : I \to \mathbb{R}$. Let X be a continuous random variable whose density f_X is zero on the complement of I. Set $Y = g(X)$.
   (a) Let g be a differentiable strictly increasing function.
      (i) Show that inverse of g exists and g^{-1} is strictly increasing on $g(I)$.
      (ii) For any $y \in \mathbb{R}$, show that $P(Y \leq y) = P(X \leq g^{-1}(y))$
      (iii) Show that Y has a density $f_Y(\cdot)$ given by

$$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y).$$

   (b) Let g be a differentiable strictly decreasing function.

(i) Show that inverse of g exists and g^{-1} is strictly decreasing on $g(I)$.

(ii) For any $y \in \mathbb{R}$, show that $P(Y \leq y) = 1 - P(X \leq g^{-1}(y))$

(iii) Show that Y has a density $f_Y(\cdot)$ given by

$$f_Y(y) = f_X(g^{-1}(y)) \left(-\frac{d}{dy} g^{-1}(y) \right).$$

7. Let $U \sim$ Uniform $(0, 1)$. Let X be a continuous random variable with a distribution function F. Extend $F : \mathbb{R} \to \mathbb{R}$ to $F : \mathbb{R} \cup \{-\infty\} \cup \{\infty\} \to \mathbb{R}$ by setting $F(\infty) = 1$ and $F(-\infty) = 0$. Define the generalised inverse of F, $G : [0, 1] \to \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ by

$$G(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\}.$$

Show that

(a) Show that for all $y \in [0, 1]$, $F(G(y)) = y$.

(b) Show that for all $x \in \mathbb{R}$ and $y \in [0, 1]$

$$F(x) \geq y \iff x \geq G(y).$$

(c) $Y = G(U)$ has the same distribution as X.

(d) $Z = F(X)$ has the same distribution as U.