Due: Thursday, March 10th 2016
Problem to be turned in: 4

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having a common distribution function F and probability density function f. Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the corresponding order statistic.
   (a) Show that the $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ has a joint probability density function given by
   $$f_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x) = \begin{cases} n!\prod_{i=1}^n f(x_i) & -\infty < x_1 < x_2 \ldots < x_n < \infty \\ 0 & \text{otherwise} \end{cases}$$
   (b) Show that for $1 \le i < j \le n$, $(X_{(i)}, X_{(j)})$ has a joint probabil ity density function given by
   $$f_{X_{(i)},X_{(j)}}(x,y) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x)f(y)[F(x)]^{i-1}[F(y)-F(x)]^{j-1-i}[1-F(y)]^{n-j},$$
   for $-\infty < x < y < \infty$.

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having a common distribution $X \sim$ Uniform $(0,1)$. Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the corresponding order statistic. Show that $\frac{X_{(1)}}{X_{(n)}}$ and $X_{(n)}$ are independent random variables.

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having a common distribution X whose probability density function is given by
   $$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
   . Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the corresponding order statistic.
   (a) Find the distribution of $\frac{X_{(i)}}{X_{(i+1)}}$ for $1 \le i < n$.
   (b) Show that $\{\frac{X_{(i)}}{X_{(i+1)}} : 1 \le i < n\}$ are mutually independent random variables.

4. Let $\{U_i\}$ be i.i.d uniform $(0,1)$ random variables and Let $N \sim$ Poisson (λ). Find the distribution of $V = \min\{U_1, U_2, \ldots, U_{N+1}\}$.

5. Let $-\infty < a < b < \infty$. Let $X_1, X_2, \ldots, X_n$ i.i.d $X \sim$ Uniform (a,b). Find the probability density function of $M = \frac{X_{(1)}+X_{(n)}}{2}$.

6. Let X_1, X_2 be i.i.d standard normal random variables. Let $U = \min\{X_1, X_2\}$. Find the distribution of $Z = U^2$.