Due: Thursday, February 18th 2016
Problem to be turned in: 4

1. Suppose (X, Y) have a joint probability density function $f : \mathbb{R}^2 \to \mathbb{R}$ given by
$$f(x, y) = Ke^{-(4x^2-16xy+2y^2)}$$
for $x, y \in \mathbb{R}$ and $K > 0$. Find $E[X], E[Y], Var[X], Var[Y]$ and $Cov[X, Y]$.

2. Suppose X_1 and X_2 are independent standard normal random variables. Show that $X_1 + X_2$ and $X_1 - X_2$ are also independent.

3. Suppose (X, Y) is Bivariate normal random variable such that $E[X] = E[Y] = 0$, $Var[X] = Var[Y] = 1$, and $\rho = 0.5$. Find the conditional distribution of $X \mid Y = 1$ and using the tables find $P(X > 1 \mid Y = 1)$.

4. Scores students in Probability 1 and Probability 2 course of B.Math (hons.) are deemed to have come from a Bivariate Normal distribution with the same mean of 30 in each course, same variance of 7 for each course,and with a correlation coefficient of 0.7. What is the conditional probability that the student gets more than 30 in Probability 2 given that the students score in Probability 1 is (a) 30, (b) 35, (c) 40 ?

5. Let X_1, X_2 be two independent normal random variables with mean 0 and variance 1. Let (Y_1, Y_2) be a bivariate normal random variable with zero means, variances 1 and correlation ρ, with $\rho^2 \neq 1$. Let f be the joint probability density function of (X_1, X_2) and g be the joint probability density function of (Y_1, Y_2). For $0 < \alpha < 1$, let (Z_1, Z_2) be a bivariate random variable with joint density given by
$$h(z_1, z_2) = \alpha g(z_1, z_2) + (1 - \alpha)f(z_1, z_2),$$
for any real numbers z_1, z_2.
   (a) Write down the exact expressions for f and g.
   (b) Verify that h is indeed a probability density function.
   (c) Show that Z_1 and Z_2 are Normal random variables by calculating their marginal densities.
   (d) Show that (Z_1, Z_2) is not a bivariate normal random variable.

6. Let X and Y be two independent Normal $(0, 1)$ random variables. Let $R = \sqrt{X^2 + Y^2}$ and $\theta = \arctan(\frac{X}{Y})$. Find the joint distribution of (R, θ).

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2882	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3079	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3290	0.3315	0.3340	0.3365	0.3389
1.0	0.3414	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3622
1.1	0.3643	0.3665	0.3687	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4083	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4193	0.4207	0.4222	0.4237	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4358	0.4370	0.4382	0.4394	0.4406	0.4418	0.4430	0.4441
1.6	0.4452	0.4463	0.4474	0.4485	0.4495	0.4505	0.4516	0.4526	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4600	0.4608	0.4617	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4679	0.4686	0.4693	0.4700	0.4706
1.9	0.4713	0.4720	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4762	0.4767
2.0	0.4773	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4813	0.4817
2.1	0.4822	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4858
2.2	0.4861	0.4865	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4914	0.4916
2.4	0.4918	0.4920	0.4923	0.4925	0.4927	0.4929	0.4931	0.4933	0.4934	0.4936
2.5	0.4938	0.4940	0.4942	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4954	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4966	0.4967	0.4968	0.4969	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4975	0.4975	0.4976	0.4977	0.4978	0.4978	0.4979	0.4980	0.4980	0.4981
2.9	0.4982	0.4982	0.4983	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4988	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4991	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4993	0.4993	0.4993
3.2	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995	0.4995
3.3	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998	0.4998	0.4998

Table 1: Normal tables evaluating : $\frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{x^2}{2}} dx$