Due: Thursday, March 31st 2016
Problem to be turned in: 7

1. Let $X \sim$ Cauchy $(0,1)$. Find the distribution of $Y = \frac{1}{X}$.
2. Let $X_1, X_2, \ldots, X_n$ be i.i.d Uniform $(0,1)$ random variables. Find the density of $Y = \prod_{i=1}^{n} X_i$.
3. Let X_1, X_2 be i.i.d. Uniform $(0,1)$ random variables. Let $Y_1 = \cos(2\pi X_2)\sqrt{-2\log(X_1)}$ and $Y_2 = \sin(2\pi X_2)\sqrt{-2\log(X_1)}$. Find the joint density of Y_1, Y_2.
4. Let X_1, X_2 be independent Exponential (λ) random variables. Find the probability density function of $Y = \frac{X_1 X_2}{(X_1+X_2)^2}$.
5. Let (X_1, X_2) be a bivariate Normal random variable. Define

$$\Sigma = \begin{bmatrix} Cov[X_1, X_1] & Cov[X_1, X_2] \\ Cov[X_1, X_2] & Cov[X_2, X_2] \end{bmatrix}$$

$$\text{and } \mu_1 = E[X_1], \mu_2 = E[X_2], \mu_{2\times 1} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}.$$

Σ is referred to as the covariance matrix of (X_1, X_2) and μ is the mean matrix of (X_1, X_2).

   (a) Compute $\det(\Sigma)$.

   (b) Show that the joint density of (X_1, X_2) can be rewritten in matrix notation as

$$g(x_1, x_2) = \frac{1}{2\pi \det(\Sigma)} \exp\left(-\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \Sigma^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right)$$

   (c) Suppose

$$A_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \eta_{2\times 1} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

   such that a_{ij} are real numbers. Suppose we define

$$Y = AX = \begin{bmatrix} a_{11}X_1 + a_{12}X_2 + \eta_1 \\ a_{21}X_1 + a_{22}X_2 + \eta_2 \end{bmatrix}.$$

   Show that (Y_1, Y_2) is also a bivariate Normal random variable, with covariance matrix $A\Sigma A^T$ and mean matrix $A\mu + \eta$.

6. Let $X_1, X_2, \ldots X_n$ be i.i.d. standard normal random variables. Let $\overline{X}$ denote the sample mean and S^2 denote the sample variance.

   (a) Show that

$$S^2 = \frac{1}{n-1}\left(\left(\sum_{i=2}^{n} X_i - \overline{X} \right)^2 + \sum_{i=2}^{n} (X_i - \overline{X})^2 \right).$$

   (b) Let $Y_1 = \overline{X}$, $Y_i = X_i - \overline{X}$ for $i = 2, 3, \ldots n$. Find the joint density of $(Y_1, Y_2, \ldots, Y_n)$.

   (c) Show that S^2 and $\overline{X}$ are independent.