Buffon Needle Problem

Due to Georges-Louis Leclerc, Comte de Buffon

- (Birth of Geometric Probability)

Q:- Considered the chances that a randomly thrown coin hits an edge of a regular mosaic paving on the floor.

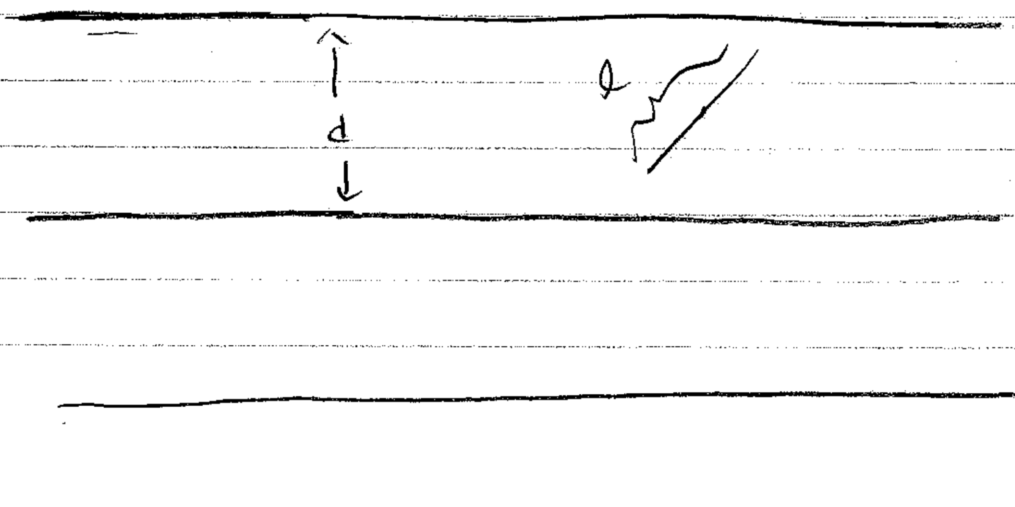
- Published results in 1777

Simplified Version – Buffon's needle

- mosaic is given by parallel lines at distance $\boxed{d}$ apart & coin is replaced by needle of length $\boxed{l}$, $l < d$.

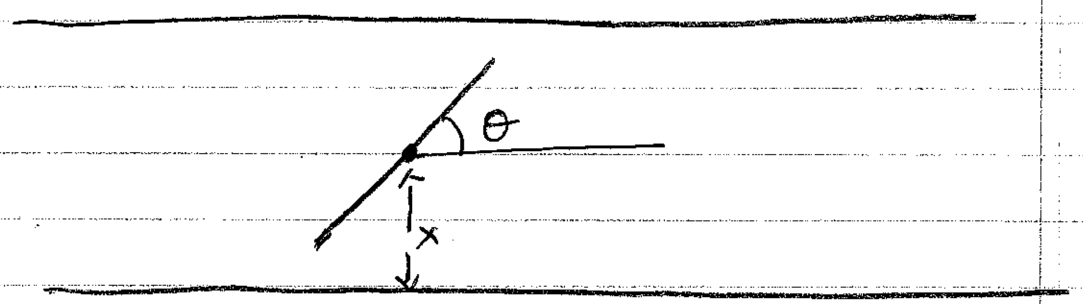
$d > 0$ $l > 0$

Q:- Drop the needle at random on the floor. What is the probability that it crosses a line?



A:- let X denote the vertical distance of the midpoint of the needle from the bottom line. (of the respective strip).

let θ denote the angle of the needle to the horizontal axis



Observation:- $X \sim \text{Uniform}(0, d)$

$\Theta \sim \text{Uniform}(0, \pi)$

X & Θ are independent.

$$f_X(x) = \begin{cases} \frac{1}{d} & 0 < x < d \\ 0 & \text{otherwise} \end{cases}$$

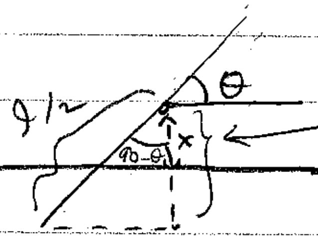
$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & 0 < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

The joint density of X and Θ is given by

$$\boxed{*} \quad f(x,\theta) = \begin{cases} \dfrac{1}{\pi d} & 0 < x < d \\ & 0 < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

Event of interest : {a line is crossed by the needle.} = E

Case 1 $\quad 0 < X < \frac{d}{2} \qquad \frac{\pi}{2} < \theta < \pi$



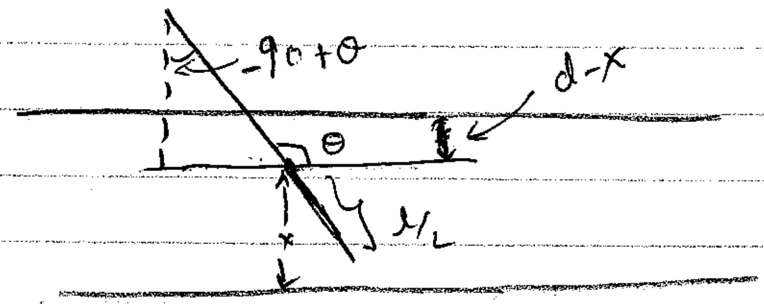
$\frac{l}{2}\cos(90-\theta) = \frac{l}{2}\sin\theta$

E − occurs $\Rightarrow \quad X < \frac{l}{2}\sin\theta$

Case 2 : $\quad 0 < X < d/2 \qquad \frac{\pi}{2} < \theta < \pi$

Ex:- $\quad$ E occurs $\Rightarrow \quad X < \frac{l}{2}\sin\theta$

Case 3 : $\quad \frac{d}{2} < X < d \qquad \frac{\pi}{2} < \theta < \pi$



E- occurs $\Rightarrow d - X < \frac{l}{2}\cos(\theta - 90) = \frac{l}{2}\sin\theta$

Case 4 :- $\frac{d}{2} < X < d \qquad 0 < \theta < \frac{\pi}{2}$

E-occurs $\Rightarrow d - X < \frac{l}{2}\sin\theta$

$\therefore$

$$E = \left\{ X < \frac{l}{2}\sin(\theta) \right\} \cup \left\{ d - X < \frac{l}{2}\sin\theta \right\}$$

$$P(E) = \iint_E f(x,\theta)\, d\theta\, dx$$

$$= \int_0^{\pi} d\theta \cdot \frac{1}{\pi} \int_0^{l/2 \sin\theta} dx \cdot \frac{1}{d}$$

$$+ \int_0^{\pi} d\theta \cdot \frac{1}{\pi} \int_{d - \frac{l}{2}\sin\theta}^{d} dx\, \frac{1}{d}$$

Used :- $\cdot$ $\boxed{*}$

$\cdot$ $\boxed{l < d} \Rightarrow \frac{l}{2}\sin\theta < \frac{d}{2}\sin\theta < \frac{d}{2}$

$\therefore \{ \underset{0<\theta<\pi}{0 < X < \tfrac{d}{2}} \} \cap \{ X < \frac{l}{2}\sin\theta \} \Leftrightarrow \{ \underset{0<\theta<\pi}{0 < X} < \frac{l}{2}\sin\theta \}$

also

- $\left\{ \frac{d}{2} < x < d \,,\, 0 < \theta < \pi \right\} \cap \left\{ d - x < \frac{l}{2} \sin\theta \right\}$

$$\Leftrightarrow \left\{ 0 < \theta < \pi \,,\, d - \frac{l}{2} \sin\theta < x < d \right\}$$

- Real Analysis: — Reordering & splitting of integrals.

$$\therefore P(E) = \frac{1}{\pi d} \left[\int_0^{\pi} d\theta \int_0^{l/2 \sin\theta} dx + \int_0^{\pi} d\theta \int_{d - \frac{l}{2}\sin\theta}^{d} dx \right]$$

$$= \frac{1}{\pi d} \left[\int_0^{\pi} d\theta \, \frac{l}{2} \sin\theta + \int_0^{\pi} d\theta \, \frac{l}{2} \sin\theta \right]$$

$$= \frac{l}{\pi d} \int_0^{\pi} \sin\theta = \frac{l}{\pi d} -\cos\theta \Big|_0^{\pi}$$

$\Rightarrow$

$$\boxed{P(E) = \frac{2l}{\pi d}}$$

Ex:- What happens when $l > d$?

Estimation of π

let $n \geq 1$. Perform Buffon's needle experiment n-times.

$$1 \leq i \leq n, \quad Y_i = \begin{cases} 1 & \text{if needle crosses a line} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad P(Y_i = 1) = P(E) = \frac{2l}{\pi d} \quad \forall \, 1 \leq i \leq n$$

$$\Rightarrow \quad Y_i \sim \text{Bernoulli}\left(\frac{2l}{\pi d}\right) \quad \& \quad E[Y_i] = \frac{2l}{\pi d} = p$$

let $S_n = \sum_{i=1}^{n} Y_i$. By weak law of large numbers, $\forall \, \varepsilon > 0$

$$P\left(\left| \frac{S_n}{n} - p \right| > \varepsilon \right) \longrightarrow 0 \quad \text{as } n \to \infty$$

$\Rightarrow$ $\frac{S_n}{n}$ is "close" to $p = \frac{2l}{\pi d}$

$\Rightarrow$ $\frac{n}{S_n} \cdot \frac{2l}{d}$ is "close" to π

$$\Rightarrow \boxed{\frac{n}{S_n} \cdot \frac{2l}{d} \text{ - estimate of } \pi}$$