independent $X \sim \text{Normal}(0,1)$

$Y \sim \text{Normal}(0,1)$

$R = \sqrt{X^2 + Y^2}$

$\theta = \arctan\left(\frac{Y}{X}\right)$

$\boxed{R}$

$X^2 \sim \text{Gamma}(1/2, 1/2)$

$Y^2 \sim \text{Gamma}(1/2, 1/2)$

$+$ X^2 & Y^2 are independent

$X^2 + Y^2 \sim \text{Exponential}(1/2)$

$R =$

$\boxed{\Theta}$

$$R = \sqrt{X^2+Y^2} \qquad P(R \le r) = \begin{cases} \frac{1}{2}\int_0^{r^2} e^{-\frac{1}{2}x}\,dx & r>0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_R(r) = \begin{cases} r e^{-\frac{r^2}{2}} & r>0 \\ 0 & \text{otherwise} \end{cases}$$

[Rayleigh]

$\boxed{\Theta}$

$$\frac{Y}{X} \sim \text{Cauchy}(1) \implies \Theta \sim \text{Uniform}(-\pi/2, \pi/2)$$

(Done in class)

$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & \theta \in (-\pi/2, \pi/2) \\ 0 & \text{otherwise} \end{cases}$$

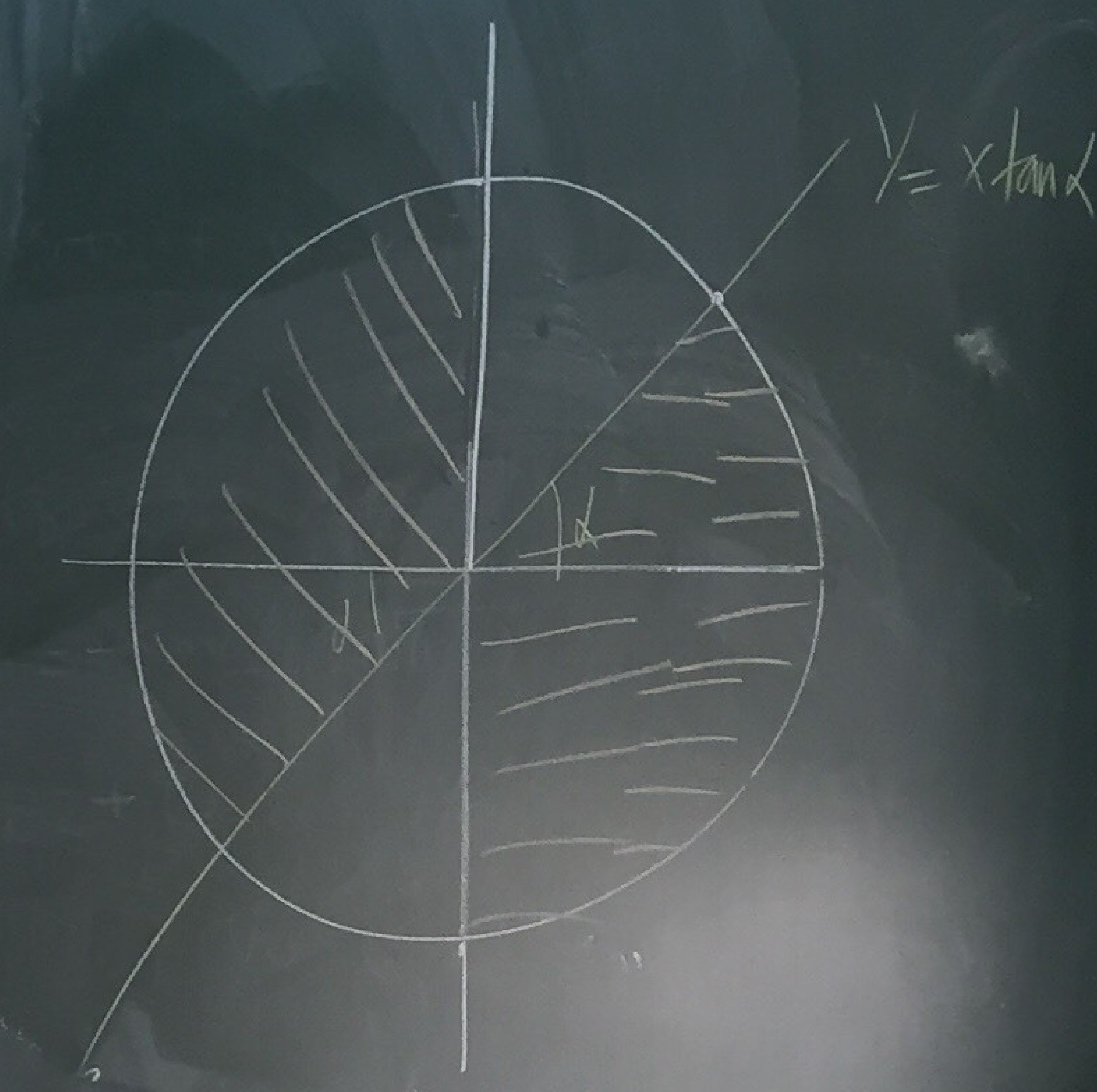
$(1/2)$

$\boxed{R, \Theta}$

$r > 0 \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\mathbb{P}(R \leq r, \ \Theta \leq \alpha) = P\left(X^2+Y^2 \leq r^2, \ \frac{Y}{X} \leq \tan(\alpha)\right)$$

$$A = \left\{(x,y) \ \middle| \ x^2+y^2 \leq r^2, \ \frac{y}{x} \leq \tan(\alpha)\right\}$$



$$P(A) = P(X^2+Y^2 \leq r^2)$$
$$- P(A^c, X^2+Y^2 \leq r^2)$$

$$P\left(x^2+y^2 \le r^2,\ \frac{y}{x} \le \tan\alpha\right)$$

$$= P(x^2+y^2 \le r^2) - \frac{2}{2\pi}\left[\int_0^{r\sin\alpha}\left(\int_0^{y\cot(\alpha)} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\,dx\right)dy + \int_{r\sin\alpha}^{r}\int_0^{\sqrt{r^2-y^2}} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\,dx\right]dy$$

$$\boxed{\begin{aligned} g(t) &= \int_0^t f(s)\,ds \\ g'(t) &= f(t) \end{aligned}}$$

$$\frac{d}{d\alpha}$$

$$h(r,\alpha) = \int_0^{r\sin\alpha}\left[\int_0^{y\cot\alpha} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\,dx\right]dy$$

$$g(r,\alpha) = \int_{r\sin\alpha}^{r}\left[\int_0^{\sqrt{r^2-y^2}} e^{-\frac{x^2}{2}} e^{-y^2/2}\,dx\right]dy$$

$$\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial r}h(r,\alpha)\right] = \frac{\partial}{\partial\alpha}\left[\sin\alpha\int_0^{r\cos\alpha} e^{-\frac{x^2}{2}} e^{-\frac{r^2\sin^2\alpha}{2}}\,dx\right]$$

$$= \frac{\partial}{\partial\alpha}\left(\sin\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}}\right)\cdot\int_0^{r\cos\alpha} e^{-\frac{x^2}{2}}dx$$

$$- r\sin^2\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}}\, e^{-\frac{r^2\cos^2\alpha}{2}}$$

$$= \left[\cos\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}} + \sin\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}}(-r^2\sin\alpha\cos\alpha)\right]\cdot\int_0^{r\cos\alpha} e^{-x^2/2}\,dx$$

$$- r\sin^2\alpha\, e^{-\frac{r^2}{2}}$$

$$\frac{\partial}{\partial \alpha}\frac{\partial}{\partial r}\left(g(r,\alpha)\right) = \frac{\partial}{\partial r}\left[\frac{\partial}{\partial \alpha} g(r,\alpha)\right]$$

$$= \frac{\partial}{\partial r}\left[(-r\cos\alpha)\int_0^{r\cos\alpha} e^{-\frac{x^2}{2}} e^{-\frac{r^2\sin^2\alpha}{2}}\,dx\right]$$

$$= \frac{\partial}{\partial r}\left[-r\cos\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}}\int_0^{r\cos\alpha} e^{-\frac{x^2}{2}}\,dx\right]$$

$$= \frac{\partial}{\partial r}\left[-r\cos\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}}\right]\int_0^{r\cos\alpha} e^{-\frac{x^2}{2}}\,dx - r\cos^2\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}} e^{-\frac{r^2\cos^2\alpha}{2}}$$

$$= \left[\begin{array}{l} -\cos\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}} \\ -r\cos\alpha\, e^{-\frac{r^2\sin^2\alpha}{2}}(-r\sin^2\alpha) \end{array}\right]\cdot \int_0^{r\cos\alpha} e^{-x^2/2}\,dx$$

$$-r\cos^2\alpha\, e^{-r^2/2}$$

$$P\left(X^2+Y^2 \le r^2, \frac{Y}{X} \le \tan\alpha\right)$$

$$= P(X^2+Y^2 \le r^2) - \frac{1}{\pi}\left[h(r,\alpha) + g(r,\alpha)\right]$$

$$f(r,\alpha) = \frac{\partial}{\partial r}\left[\frac{\partial}{\partial \alpha} P\left(X^2+Y^2 \le r^2, \frac{Y}{X} \le \tan\alpha\right)\right]$$

$$= -\frac{1}{\pi}\left[-r\sin^2\alpha\, e^{-r^2/2} - r\cos^2\alpha\, e^{-r^2/2}\right]$$

$$= \frac{1}{\pi} r e^{-r^2/2} = f_\Theta(\alpha)\, f_R(r)$$

$h(r$

$g(r,\alpha$

$$\frac{\partial}{\partial \alpha}\left[\frac{\partial}{\partial r} h(r,\alpha)\right.$$