Due: Thursday, September 24th, 2015
Problem to be turned in: 3,6

1. Suppose X is a discrete random variable which has finite expectation. Show that for any $b \in \mathbb{R}$, that $E|X-b| < \infty$ and proceed to find b_0 such that
$$E|X-b_0| = \min_{b\in\mathbb{R}} E|X-b|.$$

2. Sonia sends out invitations to eleven of her friends to join her on a hike she's planning. She knows that each of her friends has a 59% chance of deciding to join her independently of each other. Let Z denote the number of friends who join her on the hike.
   (a) What type of random variable (with what parameter) is Z?
   (b) What is the average (expected value) number of her friends that will join her on the hike?
   (c) What is the most likely number (mode) of her friends that will join her on the hike?
   (d) How do your answers to (b) and (c) change if each friend has only a 41% chance of joining her?

3. Let m and r be positive integers and let N be an integer for which $N > \max\{m, r\}$. Let X be a random variable with $X \sim \text{HyperGeo}(N, r, m)$. Find $E[X]$ and $\text{Var}[X]$.

4. Suppose X is a discrete random variable with finite variance (and thus finite expected value as well) and suppose there are two different numbers $a, b \in \mathbb{R}$ for which $P(X = a)$ and $P(X = b)$ are both positive. Prove that $Var[X] > 0$.

5. Let $X \sim \text{Geometric}(p)$ and let A be event $(X \le 3)$. Calculate $E[X|A]$ and $Var[X|A]$.

6. A standard light bulb has an average lifetime of four years with a standard deviation of one year. A Super D-Lux lightbulb has an average lifetime of eight years with a standard devaition of three years. A box contains many bulbs – 90% of which are standard bulbs and 10% of which are Super D-Lux bulbs. A bulb is selected at random from the box. What are the average and standard deviation of the lifetime of the selected bulb?

7. Let X and Y be described by the joint distribution

	$X=-1$	$X=0$	$X=1$
$Y=-1$	1/15	2/15	2/15
$Y=0$	2/15	1/15	2/15
$Y=1$	2/15	2/15	1/15

   and answer the following questions.
   (a) Calculate $E[X|Y=-1]$.
   (b) Calculate $Var[X|Y=-1]$.
   (c) Describe the distribution of $E[X|Y]$.
   (d) Describe the distribution of $Var[X|Y]$.

Midterm Solutions:

Solution 1: (a) The sample space $S = \{HH, HT, TH, TT\}$. Let $\mathcal{F}$ be the set of all subsets of S. As all outcomes in the experiment are equally likely the probability $P : \mathcal{F} \to [0, 1]$ is given by

$$P(E) = \frac{\mid E \mid}{4}.$$

It was shown in class that the above P satisfies the axioms of Probability.

(b) Clearly the Range (X)= Range (Y)= Range (Z) = $\{0, 1\}$. Further,

$$\begin{array}{c}
P(X=0) = P\{TH,TT\}) = \frac{|\{TH,TT\}|}{4} = \frac{1}{2},\ P(X=1) = P(\{HT,HH\}) = \frac{|\{HT,HH\}|}{4} = \frac{1}{2}. \\
P(Y=0) = P(\{HT,HH\}) = \frac{|\{HT,HH\}|}{4} = \frac{1}{2}, P(Y=1) = P(\{TH,TT\}) = \frac{|\{TH,TT\}|}{4} = \frac{1}{2}. \\
P(Z=0) = P(\{HT,TT\}) = \frac{|\{HT,TT\}|}{4} = \frac{1}{2},\ P(Z=1) = P(\{HH,TH\}) = \frac{|\{HH,TH\}|}{4} = \frac{1}{2}.
\end{array}$$

Therefore X, Y, Z are all Bernoulli $(\frac{1}{2})$ random variables.

(c) Let $W = X + Y$. So W can assume values $0, 1, 2$. However, on the first toss the outcome is head or tail. Therefore either $X = 1$ and $Y = 0$ or $Y = 1$ and $X = 0$. Hence $X + Y$ is the constant function 1. Therefore $P(W = 1) = 1$ and W is the constant random variable 1.

(d) Let $U = X + Z$. As X, Z are independent and Bernoulli($\frac{1}{2}$) random variables we have that U is Binomial $(2, \frac{1}{2})$. We can verify this directly as well. First U can assume values $0, 1, 2$. Further, One can also verify this,

$$\begin{aligned}
P(U=0) &= P(X+Z=0) = P(X=0, Z=0) = P(\{TT\}) = \frac{1}{4}. \\
P(U=1) &= P(X+Z=1) = P((X=1, Z=0) \cup (X=0, Z=1)) = P(\{HT, TH\}) = \frac{1}{2} \\
P(U=2) &= P(X+Z=2) = P(X=1, Z=1) = P(\{HH\}) = \frac{1}{4}.
\end{aligned}$$

(e) Clearly W and U do not have the same distribution. This is because W is a sum of dependent Bernoulli $\frac{1}{2}$ random variables but U is the sum of independent Bernoulli $\frac{1}{2}$ random variables.

Solution 2: Consider the events $A = \{$ Shyam has H1N1 virus $\}$ and $B = \{$ Shyam tested postive for H1N1 virus $\}$. We are given:

$$P(B|A) = 0.95,\ P(B|A^c) = 0.02,\ \text{ and } P(A) = 0.01.$$

Using Bayes' Theorem we have,

$$P(A^c|B) \quad = \quad \frac{P(B|A^c)P(A^c)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{(0.95)(0.99)}{(0.95)(0.01) + (0.02)(0.99)} = 0.686.$$

Solution 3: Range$(W) = \mathbb{N}$. For $n \geq 1$,

$$\begin{aligned}
P(W = n) &= P(\max X, Y\} = n) = P(\cup_{k=1}^{n-1}(X = n, Y = k) \cup_{k=1}^{n-1} (X = k, Y = n) \cup (X = n, Y = n) \\
&= \sum_{k=1}^{n-1} P(X = n, Y = k) + \sum_{k=1}^{n-1} P(X = k, Y = n) + P(X = n, Y = n) \\
&= \sum_{k=1}^{n-1} P(X = n)P(Y = k) + \sum_{k=1}^{n-1} P(X = k)P(Y = n) + P(X = n)P(Y = n) \\
&= \sum_{k=1}^{n-1} p(1-p)^{n-1}q(1-q)^{k-1} + \sum_{k=1}^{n-1} p(1-p)^{k-1}q(1-q)^{n-1} + p(1-p)^{n-1}q(1-q)^{n-1} \\
&= p(1-p)^{n-1}(1-(1-q)^{n-1}) + q(1-q)^{n-1}(1-(1-p)^{n-1}) + p(1-p)^{n-1}q(1-q)^{n-1} \\
&= p(1-p)^{n-1} + q(1-q)^{n-1} + (1-p)^{n-1}(1-q)^{n-1}(pq - p - q)
\end{aligned}$$

Solution 4. (a)Range $(Y) = \{1, 2\}$

$$P(Y = 1) = P(Y = 1, X = 0) + P(Y = 1, X = 1) + P(Y = 1, X = 2) = 0.1 + 0.2 + 0.1 = 0.4$$

and

$$P(Y = 2) = P(Y = 2, X = 0) + P(Y = 2, X = 1) + P(Y = 2, X = 2) = 0.3 + 0.2 + 0.1 = 0.6$$

(b) $P(X = 1 \mid Y = 2) = \frac{P(X=1,Y=2)}{P(Y=2)} = \frac{0.2}{0.6} = \frac{1}{3}$.

(c)

$$E[XY] = \sum_{x,y \in \text{Range}(X) \times \text{Range}(Y)} xyP(X = x, Y = y) = 1(0.2) + 2(0.1) + 2(0.2) + 4(0.1) = 1.2$$

(d)Range $(X) = \{0, 1, 2\}$

$$P(X = 0) = P(X = 0Y = 1) + P(X = 0, Y = 2) = 0.1 + 0.3 = 0.4,$$

$$P(X = 1) = P(X = 1Y = 1) + P(X = 1, Y = 2) = 0.2 + 0.2 = 0.4,$$

and

$$P(X = 2) = P(X = 2, Y = 1) + P(X = 2, Y = 2) = 0.1 + 0.1 = 0.2.$$

Therefore from (a) and above, we have

$$E[X] = 0(0.4) + 1(0.4) + 2(0.2) = 0.8, E[Y] = 1(0.4) + 2(0.6) = 1.6.$$

So, $\text{Cov}(X, Y) = 1.2 - 1.28 = -0.08$

(e) From part (d) $\text{Cov}(X, Y) = -0.08 \neq 0$, so X, Y cannot be independent.