# Solutions

1) Suppose that it takes at least 3 votes from a 4 member jury to convict a defendant. Also assume that the probability that a juror votes a guilty person innocent is 0.2, whereas the probability that a juror votes an innocent person guilty is 0.1. If each juror acts independently and 65% of defendants are guilty, find the probability that the jury renders a correct decision.

   **Solution.** Let $C$ be the event "correct verdict" and $G$ the event "guilty defendant". Then
$$\begin{aligned} P(C|\overline{G}) &= 1 - \left[\binom{4}{3}(0.1)^3(0.9) + (0.1)^4\right] &= .9963 \\ P(C|G) &= \binom{4}{3}(0.8)^3(0.2) + (0.8)^4 &= .8192 \end{aligned}$$
   so that
$$\begin{aligned} P(C) &= P(CG) + P(C\overline{G}) = P(C|G)P(G) + P(C|\overline{G})P(\overline{G}) = .8192 \times .65 + .9963 \times .35 \\ &= \boxed{.8812} \end{aligned}$$

2) In answering a multiple choice exam question, a student either knows the correct answer or randomly guesses 1 of $m$ alternatives. Let $p$ be the probability that the students knows the answer. If the student gets the correct answer, then what is the probability that the student actually knew the answer?

   **Solution.** Let
$$\begin{aligned} K &= \text{the event that the student knows the correct answer} \\ C &= \text{the event that the student picks the correct answer} \end{aligned}$$
   Then
$$P(K|C) = \frac{P(KC)}{P(C)} = \frac{P(K)}{P(KC)+P(\overline{K}C)} = \frac{P(K)}{P(K)+P(\overline{K})P(C|\overline{K})} = \frac{p}{p+(1-p)(1/m)} = \boxed{\frac{mp}{(m-1)p+1}}$$

3) A professor in an English course constructs a final exam by selecting four questions at random from a list of nine questions handed out in advance to the students. Suppose that a particular student has time to prepare answers to the first six of the nine questions. What is the probability that the student will be prepared for at least three of the four questions on the exam?

   **Solution.** The number of different exams is $\binom{9}{4}$. The number of four question exams that contain precisely three of the first six questions is $\binom{6}{3}\binom{3}{1}$. The number of four question exams that contain precisely four of the first six questions is $\binom{6}{4}$. Hence the number of four question exams that contain three or four of the first six questions is $\binom{6}{3}\binom{3}{1}+\binom{6}{4}$. The probability that the exam contains three or four of the first six questions is

$$\frac{\binom{6}{3}\binom{3}{1}+\binom{6}{4}}{\binom{9}{4}} = \frac{\frac{6\times5\times4}{3!}3+\frac{6\times5}{2!}}{\frac{9\times8\times7\times6}{4\times3\times2}} = \frac{60+15}{126} = \frac{75}{126} = \boxed{0.5952}$$

4) Let $X$ and $Y$ be random variables with a joint probability density function $f_{X,Y}(x,y) = e^{-(x+y)}$ for $0 \le x, y < \infty$. Let $Z = X/Y$.
 a) Find the cumulative probability distribution function of $Z$ and sketch its graph.
 b) Find the probability density function of $Z$ and sketch its graph.

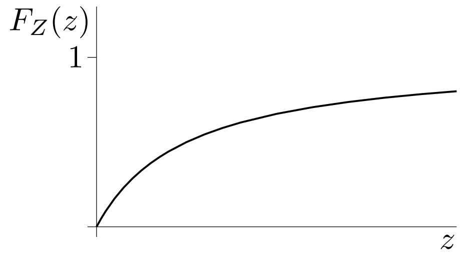
**Solution.** a) If $z < 0$, $F_Z(z) = P(Z \le z) = 0$. If $z \ge 0$,

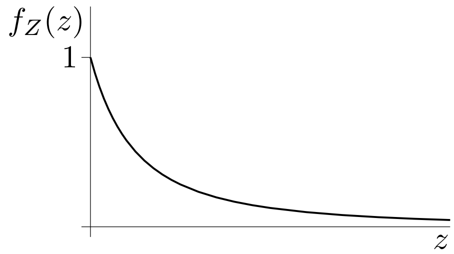
$$\begin{aligned} F_Z(z) = P(Z \le z) &= \iint_{x/y\le z} f_{X,Y}(x,y)\,dx\,dy = \int_0^\infty dy \int_0^{yz} dx\ e^{-x-y} \\ &= \int_0^\infty dy\,\Big[-e^{-y-x}\Big]_0^{yz} = \int_0^\infty \Big[e^{-y} - e^{-(1+z)y}\Big]\,dy \\ &= \Big[-e^{-y} + \tfrac{1}{1+z}e^{-(1+z)y}\Big]_0^\infty = \boxed{1-\tfrac{1}{1+z}} \end{aligned}$$

b) If $z < 0$, $f_Z(z) = \frac{d}{dz}F_Z(z) = 0$. If $z \ge 0$,

$$f_Z(z) = \tfrac{d}{dz}F_Z(z) = \tfrac{d}{dz}\big(1-\tfrac{1}{1+z}\big) = \boxed{\tfrac{1}{(1+z)^2}}$$

The graphs are




5) Let $X$ and $Y$ be independent random variables with variances $\sigma_X^2$ and $\sigma_Y^2$ respectively. For what value of $\lambda$ does the random variable $Z = \lambda X + (1-\lambda)Y$ have the smallest variance?

**Solution.** Since $\lambda X$ and $(1-\lambda)Y$ are independent,

$$\sigma_Z^2 = \sigma_{\lambda X+(1-\lambda)Y}^2 = \sigma_{\lambda X}^2 + \sigma_{(1-\lambda)Y}^2 = \lambda^2\sigma_X^2 + (1-\lambda)^2\sigma_Y^2$$

so

$$\tfrac{d}{d\lambda}\sigma_Z^2 = 2\lambda\sigma_X^2 - 2(1-\lambda)\sigma_Y^2$$

This is zero if $\lambda\big(\sigma_X^2 + \sigma_Y^2\big) - \sigma_Y^2 = 0$ or

$$\boxed{\lambda = \frac{\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}}$$

6) Let $Z$ be a standard normal random variable and $W = Z^2$.
  a) Find $E\big[e^{tW}\big]$ .
  b) Use the result of part (a) to find the variance of $W$.

**Solution.** a)

$$E\big[e^{tW}\big] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2} e^{-z^2/2}\ dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1-2t)z^2/2}\ dz$$

Make the change of variables $x = z\sqrt{1-2t}$.

$$E\big[e^{tW}\big] = \frac{1}{\sqrt{1-2t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \boxed{\frac{1}{\sqrt{1-2t}}}$$

The integral $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\ dx = 1$ because, for a standard normal random variable $X$, $P(-\infty < X < \infty) = 1$.

b) Differentiate $E\big[e^{tW}\big] = \frac{1}{\sqrt{1-2t}}$ twice with respect to t:

$$\frac{d}{dt} E\big[e^{tW}\big] = E\big[We^{tW}\big] = \frac{d}{dt} \frac{1}{\sqrt{1-2t}} = \frac{1}{(1-2t)^{3/2}}$$

$$\frac{d^2}{dt^2} E\big[e^{tW}\big] = E\big[W^2 e^{tW}\big] = \frac{d}{dt} \frac{1}{(1-2t)^{3/2}} = \frac{3}{(1-2t)^{5/2}}$$

Set $t = 0$

$$E\big[W\big] = \left.\frac{1}{(1-2t)^{3/2}}\right|_{t=0} = 1$$

$$E\big[W^2\big] = \left.\frac{3}{(1-2t)^{5/2}}\right|_{t=0} = 3$$

Hence

$$\operatorname{Var}(W) = E\big[W^2\big] - E\big[W\big]^2 = \boxed{2}$$

7) Let $X$ be a random variable with a Poisson distribution of parameter $\lambda$. Show that

$$P\big(X \text{ is even}\big) = \tfrac{1}{2}\big(1 + e^{-2\lambda}\big)$$

Hint: Look at the Taylor expansion of $e^{\lambda} + e^{-\lambda}$.

**Solution.**

$$P\big(X \text{ is even}\big) = \sum_{\substack{n=0 \\ n\ \text{even}}}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!}$$

But adding

$$e^{\lambda} = 1 + \lambda + \tfrac{\lambda^2}{2!} + \tfrac{\lambda^3}{3!} + \tfrac{\lambda^4}{4!} + \cdots$$
$$e^{-\lambda} = 1 - \lambda + \tfrac{\lambda^2}{2!} - \tfrac{\lambda^3}{3!} + \tfrac{\lambda^4}{4!} + \cdots$$

gives

$$e^{\lambda} + e^{-\lambda} = 2\Big(1 + \tfrac{\lambda^2}{2!} + \tfrac{\lambda^4}{4!} + \cdots\Big)$$

so

$$P\big(X \text{ is even}\big) = e^{-\lambda} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \tfrac{\lambda^n}{n!} = e^{-\lambda}\tfrac{1}{2}\big(e^{\lambda} + e^{-\lambda}\big) = \tfrac{1}{2}\big(1 + e^{-2\lambda}\big)$$

as desired.

8) A fair die (one die) is rolled 420 times. Approximately what is the probability that the sum of all upturned faces exceeds 1540?

**Solution.** Let $X_i$ be the number on the upturned face for roll number $i$. Then, for each $i$,

$$E[X_i] = \tfrac{1}{6}\big(1+2+3+4+5+6\big) = \tfrac{21}{6}$$
$$E[X_i^2] = \tfrac{1}{6}\big(1^2+2^2+3^2+4^2+5^2+6^2\big) = \tfrac{91}{6}$$
$$\sigma_{X_i}^2 = E[X_i^2] - E[X_i]^2 = \tfrac{35}{12}$$

By the Central Limit Theorem, $X = \sum_{i=1}^{420} X_i$ is approximately normal with mean $420 \times \frac{21}{6} = 1470$, variance $420 \times \frac{35}{12} = 1225$ and standard deviation $\sqrt{1225} = 35$. Hence, $Z = \frac{X-1470}{35}$ is approximately standard normal and

$$P(X \geq 1540) = P\big(Z \geq \tfrac{1540-1470}{35} = 2\big) = 0.5 - P\big(0 \leq Z \leq 2\big) = 0.5 - 0.4773 = \boxed{0.0227}$$