Due: Thursday, April 22, 2010

1. Suppose (X, Y) is a point chosen uniformly on the triangle $\{(x, y) : x \geq 0, y \geq 0, x + y = 4\}$. Find the conditional probability $P(Y > 1 \mid X = x)$.

2. Let X, X_n, $n \in \mathbb{N}$, be integrable random variables on a probability space,$(\Omega, \mathcal{B}, P)$ and $\mathcal{C}$ be a sub-σ algebra of $\mathcal{B}$.

   (a) Suppose $X_n, X \geq 0$ and $X_n \uparrow X$ on Ω. Then show that
   $$E(X_n \mid \mathcal{C}) \uparrow E(X \mid \mathcal{C}).$$

   (b) Suppose $X_n \to X$ such that there exists a integrable Y on $\mathcal{C}$ such that $\mid E(X_n \mid \mathcal{C}) \mid \leq Y \ \forall n$; then
   $$E(X_n \mid \mathcal{C}) \to E(X \mid \mathcal{C}).$$

3. Consider a sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $(\Omega, \mathcal{F}, P)$. An event $A \in \mathcal{B}$ is said to be *invariant* if $\exists B \in \mathcal{B}^{\mathbb{N}}$ such that for every $n \geq 1$
   $$A = \{(X_n, X_{n+1}, \ldots,) \in B\}$$
   and let $\mathcal{I} = \{A \in \mathcal{B} : A \text{ is invariant }\}$. Show that $\mathcal{I}$ a σ-algebra. It is referred to as the invariant σ-algebra

4. Consider a sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $(\Omega, \mathcal{F}, P)$. Then show that
   $$\mathcal{T} = \cap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \ldots)$$
   is a σ-algebra. It is referred to as the tail σ-algebra associated with the sequence $\{X_n\}$. Further show that $\mathcal{T} \subset \mathcal{I}$

5. Consider a sequence of independent random variables $\{X_n, n \geq 1\}$ on a probability space $(\Omega, \mathcal{F}, P)$. An event A is a tail event if it is in the tail σ-algebra. Show that $P(A) \in \{0, 1\}$.

6. Let P be a probability measure defined on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then, for all $E \in \mathcal{B}_{\mathbb{R}}$ and for all $\varepsilon > 0$, there exists an open set U and a compact set K such that $K \subseteq E \subseteq U$ and $P(U) - \varepsilon < P(E) < P(K) + \varepsilon$.

7. Let f be a non-negative measurable function defined on a Probability space $(\Omega, \mathcal{B}, \mathbb{P})$. Define
   $$\mu_f : \mathcal{B} \to [0, \infty] \text{ by } \mu_f(E) = \int_E f d\mathbb{P}.$$
   show that: (i) μ_f is a measure defined on B .

   (ii) μ_f is σ-finite if and only f is finite almost everywhere.

   (iii) $E \in \mathcal{B}, \mathbb{P}(E) = 0 \to \mu_f(E) = 0$.

8. Prove that if $f \in L^1(\Omega, \mathcal{B}, \mathbb{P})$ the following are equivalent:

   (a) $\int_E f d\mathbb{P} = 0$ for all $E \in \mathcal{B}$.

   (b) $f = 0 \ \ \mathbb{P} \ \ a.e.$