Due: Tuesday, February 11th, 2010

1. Let S be the sum of numbers obtained by rolling two biased dice with possibly different biases described by probabilities $p_1, p_2, \ldots p_6$, and $r_1, r_2 \ldots r_6$, all assumed to be non-zero.
   (a) Find the distribution of S.
   (b) Show that $P(S > 7) > P(S = 2)\frac{r_6}{r_1} + P(S = 12)\frac{r_1}{r_6}$
   (c) Deduce that no matter how the two dice are biased, the numbers $2, 7$, and 12 cannot be equally likely values for the sum. In particular, the sum cannote be uniformly distributed on the numbers from 2 to 12.
   (d) Do there exist positive integers a and b and independent non-constant random variables X and Y such that $X + Y$ has uniform distribution on the set of integers $\{a, a+1, \ldots, a+b\}$?
2. A box contains 3 red balls, 4 blue balls, and 6 green balls. Balls are drawn one by one without replacement until all red balls are drawn. Let D be the number of draws. Calculate: a) $P(D \le 9)$ b) $P(D = 9)$ and c) $E(D)$.
3. Suppose that a store buys b items in anticipation of a random demand Y, where the possible values of Y are non-negative integers y representing the number of items in demand. Suppose that each item sold brings a profit of β Rupees and each item stocked but unsold brings a loss of λ Rupees. Show that the expected loss is minimized over all b at the least integer y such that $P(Y \ge y) \ge \frac{\beta}{\beta+\lambda}$. Discuss the case $\beta = \lambda$ and $\frac{\beta}{\lambda+\beta} = \frac{k}{100}$.
4. Suppose two teams play a series of games, each producing a winner and a loser, until one team has won two more games than the other. Let G be the total number of games played. Assuming each team has chance 0.5 to win each game, independent of results of the previous games.
   (a) Find the probability function of G.
   (b) Find the expected value of G.
5. Suppose X is a random variable with density
$$f(x) = \begin{cases} cx^2(1-x), & \text{for } 0 \le x \le 1, \\ 0, & otherwise. \end{cases}$$
   Find:
   (a) The value of c.
   (b) The distribution function of X.
   (c) The conditional probability $P(X > 0.2/X < 0.5)$.
   (d) The expected value of X.
   (e) The expected value of $(X-1)^3$.
6. Let $X_1, X_2, \ldots X_n$ be independent Exponential(λ) random variables. Then show that $S_n = \sum_{k=1}^n X_k$ has a density f_n. S_n is said to be distributed as Gamma(n, λ).
7. Let $\lambda > 0$. Suppose $T = \exp(\lambda)$, then determine the distribution of $G = [T]$ the greatest integer less than or equal to T .
8. Let $r > 0, \lambda > 0$. Suppose $Y \stackrel{d}{=} \text{Gamma}(r, \lambda)$

(a) For $k > 0$, find the k-th moment of Y.

(b) If r = 1 then show that k-th moment of Y is $\frac{k!}{\lambda^k}$

9. Consider independent Bernoulli(p) trials. Let Y be a random variable that denotes the trial at which the rth Head appears.

   (a) Find the distribution of Y . Y is said to be distributed as Negative Binomial (r, p).

   (b) Calculate its mean and variance.

10. Let $\{X_k : 1 \leq k \leq n\}$ be independent continuous random variables with identical distributed as Uniform $(0, 1)$. Let

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(k)} \leq \ldots \leq X_{(n)}$$

be the increasing rearrangement of the random variables $\{X_k : 1 \leq k \leq n\}$. That is $X_{(1)}$ is the smallest of $\{X_k : 1 \leq k \leq n\}$, $X_{(2)}$ is the next smallest and so on. Show that the density of $X_{(k)}$ is given by

$$f_{(k)}(x) = \frac{1}{B(k, n-k+1)} x^{k-1}(1-x)^{n-k}, \; 0 < x < 1,$$

where $B(r, s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx$, for any $r, s > 0$. $X_{(k)}$ is referred to as the k-th order statistic and is said to have Beta$(k, n-k+1)$ distribution.