Due: Tuesday, February 2nd, 2010

1. Let $b(n, p, j)$ denote the probability of getting j successes in a Binomial(n, p) experiment. Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by:
$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}), -\infty < x < \infty.$$
Show that
$$\lim_{n\to\infty} \sqrt{npq}\, b(n, p, [np + x\sqrt{npq}]) = \phi(x).$$

2. Suppose we conduct an experiment having two outcomes ($\{S\}$ Success happens with probability p and $\{F\}$ Failure happens with probability $1 - p$ for $0 < p < 1)n$ times. Let $(\Omega, \mathcal{F}, P)$ be the corresponding probability space. Define S_n to be the number of successes in n trials. Show that
$$\lim_{n\to\infty} P(a \le \frac{S_n - np}{\sqrt{npq}} \le b) = \int_a^b \phi(x).$$

3. Show that the, m, Mode of the Binomial(n, p) distribution is given by $m = [np+p]$. Further clarify that (depending on n, p)
   (a) if np happens to be an integer then $m = np$.
   (b) if np is not an integer then m is one of the two integers to either side of np.
   (c) m may not necessarily be closest integer to np and neither is m always the integer above np nor the integer below it.

4. . Let $z > 0$. If $\Phi(z) = \int_{-\infty}^{z} \phi(x)dx$ then show that
$$1 - \Phi(z) \le \frac{\phi(z)}{z}.$$

5. Let S_{25} be the number of successes in a Binomial $(25, \frac{1}{10})$ experiment.
   (a) Find m
   (b) Find $P(S = m)$ correct upto 3 decimal places.
   (c) What is the value of the Normal approximation to $P(S = m)$?
   (d) What is the value of the Poisson approximation to $P(S = m)$?
   (e) Repeat the above if 25 is replaced by 2500. Compare the approximations given by Normal and Poisson. Repeat the same with2500 and $\frac{1}{10}$ replaced by $\frac{1}{1000}$.

6. Let X be the number of heads in three tosses of a fair coin.
   (a) Display the distribution of$X$ in a table.
   (b) Find the distribution of $|X - 1|$.

7. A box contain $2n$ balls of n different colours, with 2 of each colour. Balls are picked at random from the box with replacement until two balls of the same colour have appeared. Let X be the number of draws made. Find the distribution of X . Hint: Find $P(X > k)$

8. Let W_1 and W_2 be independent geometric random variables with parameters p_1 and p_2 . Find: $(a)P(W_1 = W_2)$ $(b)P(W_1 < W_2)$ $(c)P(W_1 > W_2)$ (d) distribution of $\min(W_1, W_2)$.

9. In $n+m$ independent Bernoulli(p) trials, let S_n be the number of successes in the first n trials, T_m the number of successes in the last m trials.

   (a) What is the distribution of S_n ?

   (b) What is the distribution of T_m ?

   (c) What is the distribution of $S_m + T_n$?

10. Suppose that the number of earthquakes X that occur in a year, anywhere in the world, is a Poisson random variable with mean ?. Suppose that the probability that any given earthquake has magnitude at least 5 on the Richter scale is p. Let N be the number of earthquakes with magnitude at least 5 in a year. Find the distribution of N .

11. At the Universal Cricket Council, five day test matches are played on a best of 5 one day games basis, that is teams A and B play until one of them has won 3 one day games. Suppose each game is won by team A with probability p, independently of all other games.

   (a) For each $g = 3, 4, 5$ find a formula in terms of p that team A wins the UCC test match in exactly g games.

   (b) Given that

      i. player A won the UCC five day test match what is the probability in terms of p that the match lasted only three games ?

      ii. B has won games 1 and 2 what is the probability in terms of p that team A wins the UCC five day test match.

      iii. Let X be a Binomial$(5, p)$ random variable. Is $P(A\text{wins}) = P(X \geq 3)$? Explain your answer intuitively as well.

      iv. Let G represent the number of games played. What is the distribution of G ? For what value of p is G independent of the winner of the series ?

12. (a) If μ is a probability measure defined on the Borel σ algebra $\mathcal{B}$ of $\mathbb{R}$, define $F : \mathbb{R} \to [0, 1]$ by $F(x) = \mu((-\infty, x])$, and verify that

   (a) F is monotonically non-decreasing - i.e. $x \leq y \longrightarrow F(x) \leq F(y)$ - and right continuous - i.e., $\lim_{y \downarrow x} F(y) = F(x)$;

   (b) F is discontinuous at x if and only if $\mu(\{x\})$¿ 0; and

   (c) $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$. The function F is referred to as the distribution function of μ.

   (d) Conversely, if $F : \mathbb{R} \to [0, 1]$ is a function satisfying (i) and (iii) above, (imitate the construction of Lebesgue measure to) show that there exists a unique probability measure μ on $\mathbb{R}$ such that $\mu((-\infty, x]) = F(x)$ for all $x \in \mathbb{R}$.

   (e) Generalise (a) and (b) above to the case of σ-finite (rather than just probability) measures.

13. Let $(\Omega, \mathcal{B}, P)$ be a probability space. Suppose

   (a) X is discrete, with range $\{x_i : i \in \mathbb{N}\}$ and $g : \mathbb{R} \to \mathbb{R}$ then $E(g(X)) = \sum_i^\infty g(x_i) P(X = x_i)$, provided $\sum_i^\infty |g(x_i)| P(X = x_i)$

   (b) X is absolutely continuous with density f and $g : \mathbb{R} \to \mathbb{R}$ then $E(g(X)) = \int g(x) f(x) dx$ provided $\int \mid g(x) \mid f(x) dx < \infty$

14. The moment generating function of a random variable X is defined to be the function

$$M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} \frac{E(X^n)}{n!} t^n.$$

   Let $I = \{t \in \mathbb{R} : M_X(t) < \infty\}$. Show that

(a) I is a (possibly degenerate) interval and $0 \in I$.

(b) $M_X(\cdot)$ is a continuous convex function on I .

(c) if 0 is an interior point of I then $E(X^k) < \infty$ for all $k \in \mathbb{N}$ (i.e. X has finite moments of all orders)

15. Let X be a random variable on the probability space $(\Omega, \mathcal{B}, P)$ with distribution $P \circ X^{-1}$. Consider the random variable $\tilde{X}$ on the probability space $(\mathbb{R}, \mathcal{B}_R, P \circ X)$ defined by $\tilde{X}(x) = x$. Then $P \circ \tilde{X}^{-1} = P \circ X^{-1}$.

16. Let $F : \mathbb{R} \to [0, 1]$ be a distribution function of a probability measure P (i.e. $F(x) = P((-\infty, x]))$. Then show that there is a random variable $X : ((0, 1], \mathcal{B}, \lambda) \to \mathbb{R}$, (where B is the Borel σ-algebra and λ is Lebesgue measure), such that $P \circ X^{-1} = P$

17. Let $X : \Omega \to \mathbb{N}$ be a random variable on a probability space $(\Omega, \mathcal{B}, P)$. Show that

$$E(X) = \sum_{n=1}^{\infty} P(X \geq n).$$

18. Show that the following are equivalent: (a) A family $\mathcal{A}_i$ of events is independent; (b) The family $\sigma(1_{\mathcal{A}_i})$ of σ-algebras is independent.

19. Let X, Y be random variables on a probability space $(\Omega, \mathcal{B}, P)$. Show that X and Y are independent if and only if $\sigma(X)$ and $\sigma(Y)$ are independent.