Due: Thursday, October 15th, 2009
Problem to be turned in: 3,5

1. Suppose we toss a biased (p) coin until we obtain the first head. Let X denote the trial at which the first head appears. Find the distribution and expectation of X.

2. Let X be a discrete random variable. Show that $E(X) < \infty$ if and only if $E(|X|) < \infty$

3. Suppose X is a random variable which has finite expectation. Show that for any $b \in \mathbb{R}$, that $E|X - b| < \infty$ and proceed to find b_0 such that

$$E|X - b_0| = \min_{b \in \mathbb{R}} E|X - b|$$

4. Let $e_n = (1 + \frac{1}{n})^n$ for $n \in \mathbb{N}$. Show that
   (a) $2 \leq e_n < 3$ and $e_n < e_{n+1}$ for all $n \geq 1$. Consequently, let $e = \lim_{n\to\infty} e_n$.
   (b) Let $s_n = \sum_{k=1}^{n} \frac{1}{k!}$. Show that $e_n \leq s_n$ and $s_n \leq \lim_{k\to\infty} e_k$. Conclude that limit of s_n exists and is e.
   (c) For $x > 0$, let $e_n(x) = (1 + \frac{x}{n})^n$ for $n \in \mathbb{N}$. Show that limit of e_n^x exists and is equal to e^x. Further, show that if $s_n(x) = \sum_{k=1}^{n} \frac{x^k}{k!}$ then limit of $s_n(x)$ exists and is also equal to e^x

5. Let f be the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x(x+1)} & x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   Verify that f is a probability mass function of a random variable X. Decide whether $E(X)$ exists or not.

6. Suppose we have a population of d elements. Let $n \leq d$. We draw a sample with replacement until exactly n distinct elements have been obtained. Let X_n denote the size of the sample required. Find $E(X_n)$.