Due: February 18th, 2008
Problems to be turned in: 1,4,8,9

1. Suppose that a store buys b items in anticipation of a random demand Y, where the possible values of Y are non-negative integers y representing the number of items in demand. Suppose that each item sold brings a profit of β Rupees and each item stocked but unsold brings a loss of λ Rupees. Show that the expected loss is minimized over all b at the least integer y such that $P(Y \geq y) \geq \frac{\beta}{\beta+\lambda}$. Discuss the case $\beta = \lambda$ and $\frac{\beta}{\lambda+\beta} = \frac{k}{100}$.
2. Let $X \stackrel{d}{=}$ Uniform$\{1, 2, \ldots, n\}$. Find $E(X)$ and $Var(X)$. Let $Y \stackrel{d}{=}$ Uniform$\{a, a+1, \ldots, a+(n-1)b\}$. Find $E(Y)$ and Var(Y) in terms of a, b and n.
3. A random variable X has expectation 10 and standard deviation 5. Find the smallest upper bound on $P(X \geq 20)$ and decide whether X can be distributed as Binomial(n, p) for some n, p.
4. Let S be the sum of numbers obtained by rolling two biased dice with possibly different biases described by probabilities $p_1, p_2, \ldots p_6$, and $r_1, r_2 \ldots r_6$, all assumed to be non-zero.
   (a) Find the distribution of S.
   (b) Show that $P(S > 7) > P(S = 2)\frac{r_6}{r_1} + P(S = 12)\frac{r_1}{r_6}$
   (c) Deduce that no matter how the two dice are biased, the numbers $2, 7$, and 12 cannot be equally likely values for the sum. In particular, the sum cannote be uniformly distributed on the numbers from 2 to 12.
   (d) Do there exist positive integers a and b and independent non-constant random variables X and Y such that $X + Y$ has uniform distribution on the set of integers $\{a, a+1, \ldots, a+b\}$?
5. Let $\lambda > 0$. Suppose $T \stackrel{d}{=} \exp(\lambda)$, then determine the distribution of $G = [T]$ the greatest integer less than or equal to T.
6. Let $r > 0, \lambda > 0$. Suppose $\gamma \stackrel{d}{=}$ Gamma(r, λ).
   (a) For $k > 0$, show that the k-th moment of γ is $\frac{1}{\lambda^k}\frac{\Gamma(r+k)}{\Gamma(r)}$, where
   $$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}, \text{ for } s > 0$$
   (b) If $r = 1$ then show that k-th moment of γ is $\frac{k!}{\lambda^k}$
7. Consider independent Bernoulli(p) trials. Let Y be a random variable that denotes the trial at which the rth Head appears.
   (a) Find the distribution of Y. Y is said to be distributed as Negative Binomial (r, p).
   (b) Calculate its mean and variance.
8. Suppose two teams play a series of games, each producing a winner and a loser, until one team has won two more games than the other. Let G be the total number of games played. Assuming each team has chance 0.5 to win each game, independent of results of the previous games. Find $E(G)$.
9. Let $\{X_k : 1 \leq k \leq n\}$ be independent continuous random variables with identical distributed as Uniform $(0, 1)$. Let
   $$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(k)} \leq \ldots \leq X_{(n)}$$
   be the increasing rearrangement of the random variables $\{X_k : 1 \leq k \leq n\}$. That is $X_{(1)}$ is the smallest of $\{X_k : 1 \leq k \leq n\}$, $X_{(2)}$ is the next smallest and so on. Show that the density of $X_{(k)}$ is given by
   $$f_{(k)}(x) = \frac{1}{B(k, n-k+1)}x^{k-1}(1-x)^{n-k}, \; 0 < x < 1,$$
   where $B(r, s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx$, for any $r, s > 0$. $X_{(k)}$ is referred to as the k-th order statistic and is said to have Beta$(k, n-k+1)$ distribution.