Due: February 9th, 2008

Problems to be turned in: 6,7,10,11

1. Let X be the number of heads in three tosses of a fair coin.
   (a) Display the distribution of X in a table.
   (b) Find the distribution of $\mid X - 1 \mid$.
2. A box contain $2n$ balls of n different colours, with 2 of each colour. Balls are picked at random from the box with replacement until two balls of the same colour have appeared. Let X be the number of draws made. Find the distribution of X. $\{$*Hint: Find* $P(X > k)$ $\}$
3. Let W_1 and W_2 be independent geometric random variables with parameters p_1 and p_2. Find:
   (a) $P(W_1 = W_2)$ (b) $P(W_1 < W_2)$ (c) $P(W_1 > W_2)$
   (d) distribution of $\min(W_1, W_2)$
4. In $n + m$ independent Bernoulli(p) trials, let S_n be the number of successes in the first n trials, T_m the number of successes in the last m trials.
   (a) What is the distribution of S_n ?
   (b) What is the distribution of T_m ?
   (c) What is the distribution of $S_m + T_n$?
5. Suppose that the number of earthquakes X that occur in a year, anywhere in the world, is a Poisson random variable with mean λ. Suppose that the probability that any given earthquake has magnitude at least 5 on the Richter scale is p. Let N be the number of earthquakes with magnitude at least 5 in a year. Find the distribution of N.
6. At the Universal Cricket Council, five day test matches are played on a "best of 5 one day games" basis, that is teams A and B play until one of them has won 3 one day games. Suppose each game is won by team A with probability p, independently of all other games.
   (a) For each $g = 3, 4, 5$, find a formula in terms of p that team A wins the UCC test match in exactly g games.
   (b) Given that
      i. player A won the UCC five day test match what is the probability in terms of p that the match lasted only three games ?
      ii. B has won games 1 and 2 what is the probability in terms of p that team A wins the UCC five day test match.
   (c) Let X be a Binomial(5,p) random variable. Is $P(\text{A wins}) = P(X \geq 3)$? Explain your answer intuitively as well.
   (d) Let G represent the number of games played. What is the distribution of G ? For what value of p is G independent of the winner of the series ?
7. (a) If μ is a probability measure defined on the Borel σ- algebra $\mathcal{B}$ of $\mathbb{R}$, define $F : \mathbb{R} \to [0, 1]$ by $F(x) = \mu((-\infty, x])$, and verify that
   (i) F is monotonically non-decreasing - i.e. $x \leq y \Rightarrow F(x) \leq F(y)$ - and right continuous - i.e., $\lim_{y \downarrow x} F(y) = F(x)$;
   (ii) F is discontinuous at x if and only if $\mu(\{x\}) > 0$; and
   (iii) $\lim_{x \to \infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0$.
   The function F is referred to as the distribution function of μ.
   (b) Conversely, if $F : \mathbb{R} \to [0, 1]$ is a function satisfying (i) and (iii) above, (imitate the construction of Lebesgue measure to) show that there exists a unique probability measure μ on $\mathbb{R}$ such that $\mu((-\infty, x]) = F(x)$ for all x in $\mathbb{R}$.
   (c) Generalise (a) and (b) above to the case of σ-finite (rather than just probability) measures.

8. Let $(\Omega, \mathcal{B}, P)$ be a probability space. Suppose
   (a) X is discrete, with range $\{x_i : i \in \mathbb{N}\}$ and $g : \mathbb{R} \to \mathbb{R}$ then $E(g(X)) = \sum_{i=1}^{\infty} g(x_i)P(X = x_i)$, provided $\sum_{i=1}^{\infty} \mid g(x_i) \mid P(X = x_i) < \infty$.
   (b) X is absolutely continuous with density f and $g : \mathbb{R} \to \mathbb{R}$ then $E(g(X)) = \int g(x)f(x)dx$ provided $\int \mid g(x) \mid f(x)dx < \infty$.
9. The **moment generating function** of a random variable X is defined to be the function $M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} \frac{E(X^n)}{n!} t^n$. Let $I = \{t \in \mathbb{R} : M_X(t) < \infty\}$. Show that
   (a) I is a (possibly degenerate) interval and $0 \in I$.
   (b) $M_X(\cdot)$ is a continuous convex function on I.
   (c) if 0 is an interior point of I then $E(X^k) < \infty$ for all $k \in \mathbb{N}$ (i.e. X has finite moments of all orders)
10. Let X be a random variable on the probability space $(\Omega, \mathcal{B}, P)$, with distribution P_X. Consider the random variable $\widetilde{X}$ on the probability space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, P_X)$ defined by $\widetilde{X}(x) = x$. Then $P_{\widetilde{X}} = P_X$.
11. Let $F : \mathbb{R} \to [0, 1]$ be a distribution function of a probability measure P (i.e. $F(x) = P((-\infty, x])$). Then show that there is a random variable $X : ((0, 1], \mathcal{B}, \lambda) \to \mathbb{R}$, (where $\mathcal{B}$ is the Borel σ-algebra and λ is Lebesgue measure),such that $P_X = P$
12. Let $X : \Omega \to \mathbb{N}$ be a random variable on a probability space $(\Omega, \mathcal{B}, P)$. Show that

$$E(X) = \sum_{n=1}^{\infty} P(X \geq n).$$

13. Show that the following are equivalent: (a) A family A_i of events is independent; (b) The family $\sigma(1_{A_i})$ of σ-algebras is independent.
14. Let X, Y be random variables on a probability space $(\Omega, \mathcal{B}, P)$. Show that X and Y are independent if and only if $\sigma(X)$ and $\sigma(Y)$ are independent.