1. Let X_n be a Markov chain on $S = \{0, 1, 2\}$ with tranistion matrix P given by
$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$
   (a) Show that the chain is irreducible.
   (b) Find the period.
   (c) Find the stationary distribution.

2. **(Random walk on $\mathbb{Z}$)** Let X_n be a Markov chain on $S = \mathbb{Z}$ with tranistion matrix P given by
$$p_{ij} = \begin{cases} p & \text{if } j = i + 1 \\ 1 - p & \text{if } j = i - 1 \\ 0 & \text{otherwise,} \end{cases}$$
where $0 < p < 1$.
   (a) Show that the chain is irreducible and every state has period 2.
   (b) Decide whether the chain is null recurrent, positive recurrent or transient.
   (c) What can you say if $p \in \{0, 1\}$?
   (d) Decide if the chain has a stationary distribution and whether it is unique.

3. Let X_n be a Markov chain with state space S and transition matrix P. Assume that X_n has period $d > 1$ and irreducible. If X_n has a stationary distribution π. Show that there exists an integer r with $0 \le r < d$ such that
$$\lim_{m\to\infty} p_{ij}^{md+r} = d\pi(j),$$
for all $j \in S$ *(Hint: Define $Y_n = X_{nd}$ for $n \ge 1$. Then Y_n is a Markov chain on S with initial distribution μ and transition matrix $Q = P^d$.*

Observe that Y_n is aperiodic. Let $j \in S$, $A_j = \{i \in S : j \leftrightarrow i\}$ (accesibility for Y_n). Show that

$$q_{ij}^m \to \frac{1}{E(T_j^Y)} = d\frac{1}{E_j(T_j)} = d\pi(j),$$

for all $i \in A_j$.

Let $r = \inf\{n \geq 1 : p_{ij}^n > 0\}$. As

$$p_{ij}^{md+r} = \sum_{k=1}^{n} f_i^{kd+r} j p_{jj}^{(m-k)d} = \sum_{k=1}^{n} f_i^{kd+r} j p_{jj}^{(m-k)d}$$

let $m \to \infty$ and complete the proof.)

4. Let $i \in S$ be a positive recurrent state. Let $\mu : S \to [0, \infty)$ by

$$\mu(j) = \sum_{k=0}^{\infty} P_i(X_k = j, T_i > k),$$

for $j \in S$. Show that $\pi : S \to [0, 1]$ given by

$$\pi(j) = \frac{\mu(j)}{E_i(T_i)},$$

for $j \in S$ defines a stationary distribution for the chain X_n.

5. Let X_n be an irreducible Markov Chain on S with transition matrix P and stationary distribution π. Let $\mid S \mid$ be finite. Suppose $p_{ii} = 0$ for all $i \in S$. Consider a matrix Q given by

$$q_{ij} = \begin{cases} 1 - q_i & \text{if } i = j \\ q_i p_{ij} & \text{if } i \neq j \end{cases}$$

   (a) Show that Q is also irreducible.
   (b) Decide whether $\lim_{n\to\infty} q_{ij}^n$ exists and if it does find the limit.