1. Consider the queing chain defined in class. Let $\xi, \xi_1, \xi_2, \dots$ be i.i.d random variables such that

$$P(\xi = k) = p_k, \qquad k = 0, 1, 2, \dots, \tag{1}$$

with $\sum_{k=0}^{\infty} p_k = 1$ (think of ξ_i as the number of people arriving in time unit i). Let X_0 be the number of people in the queue at time 0. Then the number of people in the queue at time $n \geq 1$ can be described by $X_n = \max\{X_{n-1} - 1, 0\} + \xi_n$. It can be verified that X_n is a Markov chain on state space $S = \{0, 1, 2, \dots\}$ with some initial distribution μ and transition matrix P given by

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & \cdots \\ p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & \cdots \\ 0 & p_0 & p_1 & p_2 & p_3 & p_4 & \cdots \\ 0 & 0 & p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & 0 & 0 & p_0 & p_1 & p_2 & \cdots \\ \vdots & & & \cdots & & \cdots & \cdots \end{pmatrix}.$$

Prove the assertion that the queing chain is irreducible if $p_0 > 0$ and if there exists a $k > 0$ such that $p_k > 0$.

2. Let X_n be a Markov chain on $S = \{1, 2, 3, 4, 5\}$ with the transition probability matrix

$$P = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(i) Find the set of transient states, and the irreducible closed set(s) of recurrent states. (ii) Find the probability of eventual absorption in the irreducible closed set(s) of recurrent states

3. Consider the Markov chain X_n on $S = \{0, 1, 2, 3, 4, 5, 6\}$ with initial distribution μ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Decompose the state space S into transient and closed communicating class of recurrent states.