1. Let X_n be a Markov chain on state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

   (a) Sketch the graph induced by this Markov chain on the vertex set S.

   (b) Determine the closed communicating classes and their periodicity.

2. **(Random walk on a Circle)** Let $0 < p < 1$, S be $\{0, 1, 2, \ldots, L\}$, $\mu : S \times S \to [0, 1]$, given by

$$\mu_{i,j} = \begin{cases} p & \text{if } j = i + 1, i \neq L, \text{ or } j = 0, i = L \\ 1 - p & \text{if } j = i - 1, i \neq 0, \text{ or } j = L, i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

   and $E = \{\{i, j\} : \mu_{i,j} > 0\}$. Consider the Markov chain $\{X_n\}$, as the random walk on the weighted graph (S, E, μ).

   (a) Write down the transition probability matrix, when $L = 6$.

   (b) Sketch the graph induced by this Markov chain on the vertex set S.

   (c) For general L, show that the chain is irreducible.

3. Consider the Markov chain $\{X_n\}_{n \geq 0}$ on $S = \{0, 1, 2, 3, 4, 5, 6\}$ with the

following transition matrix:

$$P = \begin{pmatrix} 0 & \frac{3}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

For any subset $A \subset [0,1]$. let $h^A : S \to [0,1]$ be given by $h^A(i) = \mathbb{P}_i(T^A < \infty)$

(a) Compute h^A when $A = \{6\}$

(b) Compute h^A when $A = \{3\}$

4. **(Gambler's Ruin Chain)** Let S be $\{0, 1, 2, \ldots, L\}$, the transition matrix $P = [p_{ij}]$ be given by

$$p_{ij} = \begin{cases} 1 & \text{if } j = 0, i = 0, \text{ and } j = L, i = L. \\ p & \text{if } j = i+1, i \neq L, \\ 1-p & \text{if } j = i-1, i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $h : S \to [0,1]$ given by $h(i) = \mathbb{P}_i(T^{\{0\}} < T^{\{L\}})$.