1. For all $i \in \mathbb{Z}$, let $\lambda_i \geq 0, \mu_i \geq 0$ Consider a a continous time Markov chain $\{X_t\}_{t\geq 0}$ on $\mathbb{Z}$ with generator matrix Q given by

$$q_{ij} = \begin{cases} \lambda_i & \text{if } j = i+1 \\ \mu_i & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

   Describe the backward equations for the above chain that the elements of the transition semigroup $P(t)$ has to satisfy.

2. Let $\{X_t\}_{t\geq 0}$ be a continuous time Markov Chain on S with generator matrix Q. Let $A \subset S$ and

$$T^A = \inf\{t \geq 0 : X_t \in A\}.$$

   (a) Let $\{Y_n\}_{n\geq 0}$ be the jump chain associated to X and $T_Y^A = \min\{k \geq 0 : Y_k \in A\}$. Show that

$$\{T^A < \infty\} = \{T_Y^A < \infty\}$$

   (b) Let $h^A : S \to [0,1]$ be given by $h^A(i) = P_i(T^A < \infty)$, show that h^A is the miminal non-negative solution to the system of linear equations

$$\begin{array}{ll} h^A(i) = 1 & \text{for } i \in A, \\ \sum_{j\in I} q_{ij} h^A(j) = 0 & \text{for } i \notin A \end{array}$$

   (c) Assume $q_i > 0$ for all $i \notin A$. Let $k^A : S \to [0,1]$ be given by $k^A(i) = E_i(T^A)$, show that k^A is the miminal non-negative solution to the system of linear equations

$$\begin{array}{ll} k^A(i) = 0 & \text{for } i \in A, \\ \sum_{j\in I} q_{ij} k^A(j) = -1 & \text{for } i \notin A \end{array}$$

3. Consider the Markov chain on $\{1, 2, 3, 4\}$ with generator matrix

$$Q = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{-1}{2} & 0 & \frac{1}{4} \\ \frac{1}{6} & 0 & \frac{-1}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Calculate

(a) the probability of hitting 3 starting from 1

(b) the expected time to hit 4 starting from 1.

4. (**M \M \ 1 Queue**) Let us consider a single queue at a ticket counter. It takes a random time to serve one customer, this service time is distributed as Exponential(μ). The customers arrive according to a Poisson process of rate λ, i.e the interarrival times between customers is distributed as Exponential (λ). Let X_t denote the state of the system at time t.

   (a) Find the generator matrix Q and the jump matrix π of the above chain.

   (b) Find the distribution of the holding times, S_i at the state $i \in \{0\} \cup \mathbb{N}$.

   (c) If $\lambda < \mu$, find the stationary distribution of the jump chain of X.

   (d) Show that if $\lambda > \mu$ the queue length explodes as $t \to \infty$.

5. Let $\{X_t\}_{t \geq 0}$ be a continuous time Markov Chain on S with generator matrix Q. Let P be the transition semigroup of X and let Y be the associated jump chain of X. Suppose for dicrete time Markov chain Y, $i \to j$ then show that $p_{ij}(t) > 0$ for all $t > 0$.

6. Let $\{X_t\}_{t \geq 0}$ be a continuous time Markov Chain on $\mathbb{Z}$ with generator matrix Q. Let $P(t)$ be its transition semigroup. Show that

$$p_{ij}(t+s) = \sum_{k \in \mathbb{Z}} p_{ik}(t) p_{kj}(s),$$

   for all $i, j \in \mathbb{Z}$.

7. Claims to an insurance company arrive according to a Poisson process X with rate λ. Let us denote the arrival times as $\{W_i\}_{i \geq 1}$. Let $C_i > 0$ (indepdenent of W_i) be the instantaneous claim payment made by the company at time W_i. Suppose

$$S(t) = \sum_{i=1}^{X_t} e^{-rW_i} C_i,$$

   denote the discounted value of the cummulative claim amount over the period $[0, t]$ and $r > 0$. Find $E[S(t)]$ for $t > 0$..