I) $Y_m = X_{n+m} \mid X_n = i \; ; \; n, m \geq 0$

WTS) Y_m is a Markov Chain with initial distribution δ_i & transition Matrix P

1) $P(Y_0 = j) = P(X_n = j \mid X_n = i) = \dfrac{P(X_n = i, X_n = j)}{P(X_n = i)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w} \end{cases}$

$\therefore$ Initial distribution is δ_i

II) $P(Y_{k+1} = i_2 \mid Y_k = i_1) = \dfrac{P(Y_{k+1} = i_2, Y_k = i_1)}{P(Y_k = i_1)} = \dfrac{P(X_{k+1+n} = i_2, X_{k+n} = i_1 / X_n = i)}{P(X_{k+n} = i_1 \mid X_n = i)}$

$= \dfrac{P(X_{k+1+n} = i_2, X_{k+n} = i_1, X_n = i)}{P(X_{k+n} = i_1, X_n = i)}$

$\dfrac{\left(\sum_{\substack{j_k \in S \\ k \in \{1,2,\dots,k-1\}}} \mu(i) P_{i j_1} P_{j_1 j_2} \cdots P_{j_{k-2} j_{k-1}} P_{j_{k-1} i_1}\right) P_{i_1 i_2}}{\left(\sum_{\substack{j_k \in S \\ k \in \{1,2,\dots,k-1\}}} \mu(i) P_{i j_1} P_{j_1 j_2} \cdots P_{j_{k-2} j_{k-1}} P_{j_{k-1} i_1}\right)}$

$= P_{i_1 i_2} \Rightarrow$ Transition Matrix of $\{Y_m\}_{m \geq 0}$ is P

For Memoryless Property

Let $i_0 = i$, so $P(Y_0 = i) = 1$

$$P(Y_m = j \mid Y_0 = i_0, \ldots, Y_{m-1} = i_{m-1}) = \frac{P(Y_m = j, Y_0 = i_0, \ldots, Y_{m-1} = i_{m-1})}{P(Y_0 = i_0, \ldots, Y_{m-1} = i_{m-1})} = \frac{\delta_i(i_0) p_{i_0 i_1} \cdots p_{i_{m-1} j}}{\delta_i(i_0) p_{i_0 i_1} \cdots p_{i_{m-2} i_{m-1}}}$$

{ Shown that δ_i is initial distribution }

$$= p_{i_{n-1} j} = L.H.S$$

$$P(Y_m = j \mid Y_{m-1} = i_{m-1}) = \frac{P(Y_m = j, Y_{m-1} = i_{m-1})}{P(Y_{m-1} = i_{m-1})} = \frac{\sum\limits_{\substack{i_k \in S \\ k \in \{1,2,\ldots,m-2\}}} P(Y_0 = i_0, Y_1 = i_1, \ldots, Y_{m-2} = i_{m-2}, Y_{m-1} = i_{m-1}, Y_m = j)}{\sum\limits_{\substack{i_k \in S \\ k \in \{1,2,\ldots,m-2\}}} P(Y_0 = i_0, Y_1 = i_1, Y_2 = i_2, \ldots, Y_{m-2} = i_{m-2}, Y_{m-1} = i_{m-1})}$$

{ $i_0 = i$ w.p. 1 }

$$\frac{\cancel{\left(\sum\limits_{\substack{i_k \in S \\ k \in \{1,2,\ldots,m-2\}}} \delta_i(i) p_{i i_1} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{m-2} i_{m-1}}\right)} p_{i_{m-1} j}}{\cancel{\left(\sum\limits_{\substack{i_k \in S \\ k \in \{1,2,\ldots,m-2\}}} \delta_i(i) p_{i i_1} p_{i_1 i_2} \cdots p_{i_{m-2} i_{m-1}}\right)}} = p_{i_{m-1} j} = RHS$$

$LHS = RHS \Rightarrow P(Y_m = j \mid Y_0 = i_0, \ldots, Y_{m-1} = i_{m-1})$

$= P(Y_m = j \mid Y_{m-1} = i_{m-1}) = p_{i_{m-1} j}$

$\Rightarrow$ It satisfies Memoryless property

$\therefore$ Markov chain with initial distribution δ_i

& Transition Matrix P

$\blacksquare$

2. $\{X_n\}_{n\geq 0}$ Markov chain on state space $S = \{1,2,3\}$

Transition matrix: $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

WTS that $p^n_{11} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left\{ \frac{4}{5}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{5}\sin\left(\frac{n\pi}{2}\right) \right\}$

Let $n=2$ then RHS $= \frac{1}{5} + \left(\frac{1}{2}\right)^2 \left\{ \frac{4}{5}\cos(\pi) - \frac{2}{5}\sin(\pi) \right\} = \frac{1}{5} + \frac{1}{4}\left\{ \frac{4}{5} \times (-1) \right\} = \underline{\underline{0}}$

But LHS $= p^2_{11} = 0\times 0 + 0\times 0 + 1/2 \times 1 = 1/2$

So the expression is incorrect

3. $0<p<1$ $\quad n\geq 1$ $\quad Z_0, Z_1, \ldots$ i.i.d. Bernoulli(p)

$S_0 = 0$, $S_1 = Z_1$, $S_n = Z_1+\cdots+Z_n$

(a) $X_n = Z_n$. Claim: $\{X_n\}_{n\geq 0}$ is a Markov Chain with state space $S=\{0,1\}$ transition matrix $P = \begin{pmatrix}1-p & p\\ 1-p & p\end{pmatrix}$ and initial distribution $\mu \sim$ Bernoulli(p)

Proof: we have $\mathbb{P}(X_0=i_0, X_1=i_1,\ldots,X_n=i_n) = \mathbb{P}(Z_0=i_0, Z_1=i_1,\ldots,Z_n=i_n)$

$$\overset{\text{independence of } Z_i\text{'s}}{=} \mathbb{P}(Z_0=i_0)\cdots\mathbb{P}(Z_n=i_n) = (1-p)^{\#\{i_k=0\}}\, p^{\#\{i_k=1\}}$$

$$= \mu(\{i_0\})\, P_{i_0 i_1}\cdots P_{i_{n-1} i_n}, \text{ because, } \mu(\{i_0\}) = \begin{cases}1-p & \text{if } i_0=0\\ p & \text{if } i_0=1\end{cases}$$

$$\& \; P_{i_k i_{k+1}} = \begin{cases} p & \text{if } i_{k+1}=1 \\ 1-p & \text{if } i_{k+1}=0\end{cases}$$

$\{X_n\}$ is a Markov Chain with the said properties.

(b) X_n
Clai
$P=$
Proof:
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In any ot
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{Shown that δ_i is initial distribution}

$$= p_{i_{n-1} j} = LHS$$

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{$i_0 = i$ w.p. 1}

$$\cancel{\frac{\left(\sum\limits_{\substack{i_k \in S \\ k \in \{1,2,\ldots,m-2\}}} \delta_i(i)\, p_{i i_1} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{m-2} i_{m-1}}\right) p_{i_{m-1} j}}{\left(\sum\limits_{\substack{i_k \in S \\ k \in \{1,2,\ldots,m-2\}}} \delta_i(i)\, p_{i i_1} p_{i_1 i_2} \cdots p_{i_{m-2} i_{m-1}}\right)}}$$

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$\boxed{///}$

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3 $0 < p < 1$ $n \geq 1$ $Z_0, Z_1, \ldots$ i.i.d Bernoulli(p)

$S_0 = 0$, $S_1 = Z_1$, $S_n = Z_1 + \ldots + Z_n$

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Proof: we have $\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \mathbb{P}(Z_0 = i_0, Z_1 = i_1, \ldots, Z_n = i_n)$

$\overset{\text{independence of } Z_i\text{'s}}{=} \mathbb{P}(Z_0 = i_0) \ldots \mathbb{P}(Z_n = i_n) = (1-p)^{\#\{i_k = 0\}} p^{\#\{i_k = 1\}}$

$= \mu(\{i_0\}) P_{i_0 i_1} \ldots P_{i_{n-1} i_n}$, because, $\mu(\{i_0\}) = \begin{cases} 1-p & \text{if } i_0 = 0 \\ p & \text{if } i_0 = 1 \end{cases}$

$\& \; P_{i_k i_{k+1}} = \begin{cases} p & \text{if } i_{k+1} = 1 \\ 1-p & \text{if } i_{k+1} = 0 \end{cases}$

$\{X_n\}$ is a Markov Chain with the said properties.

(b) X_n ... Clai... $P =$ ... Proof: ... Then ... In any ot... sides a...

(b) $X_n = S_n = Z_1 + \dots + Z_n$ Assume $X_0 = S_0 = 0$

Claim $\{X_n\}$ is a Markov Chain with state space $S = \mathbb{Z}_{\geq 0}$, transition matrix

$$P = \begin{pmatrix} 1-p & 0 & 0 & \cdots \\ p & 1-p & 0 & \cdots \\ 0 & p & 1-p & \cdots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix} = (p_{ij}) \quad \text{where } p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i \end{cases} \;\&\; p_{ij} = 0 \text{ otherwise}$$

& initial distribution $\mu \sim \delta_0$.

Proof: Assume $0 = i_0 \le i_1 \le \dots \le i_n \le n$, $i_k \le k$, $i_{k+1} - i_k \in \{0,1\}$. $i_0 = 0$

Then, $\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = \mathbb{P}(Z_0 = i_0, Z_1 = i_1 - i_0, \dots, Z_n = i_n - i_{n-1})$

$\overset{\text{independence \& } Z_i\text{'s}}{=} \mathbb{P}(Z_0 = i_0)\mathbb{P}(Z_1 = i_1) \cdots \mathbb{P}(Z_n = i_n - i_{n-1}) = p^{\#\{j_k = 1\}} (1-p)^{\#\{j_k = 0\}}$ where $j_k = i_k - i_{k-1}$

$= \mu(\{0\}) \cdot p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$ by the definition of p_{ij}

In any other case, both the left & right hand sides are zero. Hence, the equation holds for each $i_k \in S$. $\therefore \{X_n\}$ is a Markov Chain with said properties.