Solution 1: In the above Ordinary Differential Equation (ODE), the coefficients are constant. We first check if $x : [0, \infty) \to \mathbb{R}$ given by $x(t) = e^{\lambda t}$, for some $\lambda \in \mathbb{R}$ is a solution (why ?). Clearly x is twice differentiable and if it is a solution then

$$\begin{aligned} e^{\lambda t}(\lambda^2 - 4\lambda + 13) &= 0, \ \forall\, t > 0 \\ \text{if and only if} & \\ \lambda^2 - 4\lambda + 13 &= 0 \end{aligned}$$

This would imply that $\lambda = 2 + 3i$ or $2 - 3i$ and is not real. Consequently this provides two candidates (non-solutions):

$$x_1(t) = e^{2t}(\cos(3t) + i\sin(3t)) \text{ or } x_1(t) = e^{2t}(\cos(3t) - i\sin(3t))$$

However, we can try

$$y_1(t) = x_1(t) + x_2(t) = 2e^{2t}\cos(3t)$$

and

$$y_2(t) = x_1(t) - x_2(t) = 2e^{2t}\sin(3t)).$$

as candidates. We proceed to check if they are solutions and linearly independent. Observer

$$\begin{aligned} \frac{dy_1}{dt}(t) &= 2y_1(t) - 6e^{2t}\sin(3t) = 4e^{2t}\cos(3t) - 6e^{2t}\sin(3t), \\ \frac{d^2y_1}{dt^2}(t) &= 2(2y_1(t) - 6e^{2t}\sin(3t)) - 18e^{2t}\cos(3t) - 12e^{2t}\sin(3t) = -24e^{2t}\sin(3t) - 10e^{2t}\cos(3t) \end{aligned}$$

$$\begin{aligned} \frac{d^2y_1}{dt^2}(t) - 4\frac{dy_1}{dt}(t) + 13y_1(t) = &-24e^{2t}\sin(3t) - 10e^{2t}\cos(3t) \\ &-4(4e^{2t}\cos(3t) - 6e^{2t}\sin(3t)) + 26e^{2t}\cos(3t) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \frac{dy_2}{dt}(t) &= 2y_2(t) + 6e^{2t}\cos(3t) = 4e^{2t}\sin(3t) + 6e^{2t}\cos(3t), \\ \frac{d^2y_2}{dt^2}(t) &= 2(2y_2(t) + 6e^{2t}\cos 3t)) + 12e^{2t}\cos(3t) - 18e^{2t}\sin(3t) = -10e^{2t}\sin(3t) + 24e^{2t}\cos(3t) \end{aligned}$$

$$\begin{aligned} \frac{d^2y_2}{dt^2}(t) - 4\frac{dy_2}{dt}(t) + 13y_1(t) = &-10e^{2t}\sin(3t) + 24e^{2t}\cos(3t) \\ &-4(4e^{2t}\sin(3t) + 6e^{2t}\sin(3t)) + 26e^{2t}\sin(3t) \\ &= 0 \end{aligned}$$

Hence y_1 and y_2 are solutions to the ODE. Further,

$$ay_1(t) + by_2(t) = 0\,\forall\, t \geq 0$$

Evaluating the above at $t = 0$ and $t = \frac{\pi}{2}$ we get that $a = b = 0$. Therefore they are linearly independent solutions. We know by Theorem 0.1 the solution set is two-dimensional. Hence any general solution is of the form

$$y(t) = ay_1(t) + by_2(t) = e^{2t}(2a\cos(3t) + 2b\sin(3t)),$$

where $a, b \in \mathbb{R}$ and $t \geq 0$.

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Solution 2: Let $x : [0, \infty) \to \mathbb{R}$ be the amount owed by Munuram. The initial avalue problem satisfied by x is given by

$$\frac{dx}{dt} = \frac{x}{10} - 1200, \ \forall\, t > 0$$

with $x(0) = 10000$.

Applying Theorem 0.2 with $p(t) = -\frac{1}{10}$ for all $t \geq 0$ and $q(t) = 1200$ for all $t \geq 0$, we know that the unique solution to the above initial value problem is given by

$$x(t) = e^{-\frac{t}{10}} \left[10000 - \int_0^t e^{\frac{s}{10}}(-1200)ds \right] = 12000 - 2000e^{\frac{t}{10}}.$$

We need to find T such that $x(T) = 0$. This would imply

$$12000 - 2000e^{\frac{T}{10}} = 0, \quad \text{and } T = 10\ln 6.$$

∎

Solution 3 (a) Following the method to solve Bernoulli equations, assume that a solution y exists and is non-zero for all $t > 0$. Let $y(0) = \alpha > 0$. Define $z : [0, \infty) \to \mathbb{R}$ such that $z(t) = \frac{1}{[y(t)]^2}$ for all $t \geq 0$. Then, z satisfies the initial value problem given by

$$\frac{dz}{dt} = \frac{-2}{[y(t)]^3} \frac{dy}{dt} = \frac{-2}{[y(t)]^3}(y(9 - y^2) = -18z + 2,$$

for all $t > 0$ and $z(0) = \frac{1}{\alpha^2} > 0$. By Theorem 0.2, we know that there is a unique solution given by

$$z(t) = e^{-18t} \left[\frac{1}{\alpha^2} - \int_0^t e^{18s}(-2)ds \right] = \frac{e^{-18t}}{\alpha^2} + \frac{1}{9}(1 - e^{-18t}),$$

for all $t \geq 0$. Clearly by inspection as $0 < e^{-18t} \leq 1$ for all $t \geq 0$, we have that $z(t) > 0$ for all $t \geq 0$. Then by Theorem 0.3 we know that $y(t) = \frac{1}{\sqrt{z(t)}}, t \geq 0$ (What if $y(0) < 0$?) is the unique solution to the initial value problem

$$\frac{dy}{dt} = y(9 - y^2), \; t > 0$$

and $y(0) = \alpha > 0$. Therefore

$$y(t) = \left(\frac{e^{-18t}}{\alpha^2} + \frac{1}{9}(1 - e^{-18t}) \right)^{-\frac{1}{2}} = \frac{3\alpha}{\sqrt{\alpha^2 + (9 - \alpha^2)e^{-18t}}}$$

for $t \geq 0$, is the unique solution.

Observe that: if $\alpha = 3$ then $y(t) = 3$ for all $t \geq 0$; if $\alpha > 3$ then $3 \leq y(t) \leq \alpha$ for $t \geq 0$; and if $\alpha < 3$ then $\alpha \leq y(t) \leq 3$ for $t \geq 0$. Therefore there exists positive constants c_1, c_2 (depending on α) such that

$$c_1 \leq z(t) \leq c_2 \text{ and } c_1 \leq y(t) \leq c_2, \text{ for all } t \geq 0.$$

So,

$$\begin{aligned} |\, y(t) - 3 \,| &\leq c_3 \,|\, [y(t)]^2 - 9 \,| \\ &= c_3 \,|\, \frac{1}{z(t)} - 9 \,| \\ &\leq c_4 \,|\, z(t) - \frac{1}{9} \,| \\ &\leq c_4 \,|\, \frac{1}{\alpha^2} - \frac{1}{9} \,|\, e^{-18t} \end{aligned}$$

for some positive constant $c_3$and c_4.

Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that $N \geq 18 \ln(\frac{1}{\epsilon})$. Then for all $t > N$, we have $e^{-18t} < \epsilon$. Therefore for $t > N$, we have

$$\begin{aligned} \mid y(t) - 3 \mid & \leq c_4 \mid \frac{1}{\alpha^2} - \frac{1}{9} \mid e^{-18t} \\ & < c_4 \mid \frac{1}{\alpha^2} - \frac{1}{9} \mid \epsilon. \end{aligned}$$

As $c_4 > 0, \alpha > 0$ are constants and $\epsilon > 0$ is arbitrary we can conclude that

$$\lim_{t \to \infty} y(t) = 3.$$

∎

(b) Let $y_1(t) = 0$ for all $t \geq 0$. Clearly $y_1(t) = \int_0^t ds y_1(s)(9 - y_1(s)^2)$ for all $t \geq 0$. Therefore the y_1 is a solution to the iniital value problem with $y(0) = 0$. Suppose y_2 is another solution to the initial value problem such that $y_2(t_0) = \beta > 0$ for some $t_0 > 0$. Let

$$a = \sup\{0 \leq s < t_0 : y(s) = 0\}.$$

By continuity of y_2 we have $y_2(a) = 0, a < t_0$, and $\exists\, \epsilon > 0$ such that

$$y_2(s) > 0 \text{ for } s \in (a, t_0 + \epsilon).$$

Define $z_2 : (a, t_0 + \epsilon) \to \mathbb{R}$ given by

$$z_z(s) = \frac{1}{[y_2(s)]^2} \text{ for } s \in (a, t_0 + \epsilon).$$

Note z_2 solves the initial value problem

$$\begin{aligned} \frac{dz}{dt} &= -18z + 2 \; s \in (a, t_0 + \epsilon) \\ z(t_0) &= \frac{1}{\beta^2} \end{aligned}$$

Using Theorem 0.4 it is clear that z_2 agrees with unique solution of the above intial value problem w. We now observe for $s \in (a, t_0 + \epsilon)$

$$y_2(s) = \frac{1}{\sqrt{z_2(s)}} = \frac{1}{\sqrt{w(s)}}.$$

By definition of a solution $\lim_{s \downarrow a} w(s) = w(a) \in [0, \infty)$. However this will imply

$$0 = y_2(a) = \lim_{s \downarrow a} y_2(s) = \lim_{s \downarrow a} \frac{1}{\sqrt{w(s)}} = \frac{1}{\sqrt{w(a)}} \in (0, \infty) \cup \{\infty\}.$$

Hence we have shown that $y_1(t) = 0$ for all $t \geq 0$ is the unique solution to the initial value problem. (How to handle $\beta < 0$?)

∎

Definition 0.1 *Let (α, β) be a bounded open interval in $\mathbb{R}$. We say $x : [\alpha, \beta] \to \mathbb{R}$ is a solution to*

$$h(t, x, x', x'', \ldots x^{(n)}) = 0, \tag{1}$$

for some $h : [\alpha, \beta] \times \mathbb{R}^n$ if x satisfies (1) for all $t \in (\alpha, \beta)$ and $x \in C([\alpha, \beta])$

Theorem 0.1 *Let $p, q \in C([0, \infty))$. The set of solutions to the second order linear homogeneous equation*

$$x'' + p(t)x'(t) + q(t)x(t) = 0,$$

for $t > 0$ is a 2-dimensional real vector space.

Theorem 0.2 *Let $p, q \in C[0, \infty)$. Then the general solution to the first order linear ODE given by*

$$x' + px + q = 0,$$

for $t > 0$ is given by

$$x(t) = e^{-\int_0^t p(s)ds}[K - \int_0^t e^{\int_0^s p(r)dr} q(s)ds], \; t > 0$$

with $K \in \mathbb{R}$.

Theorem 0.3 *Let $a > 0, p, q \in C[0, \infty)$. Assume that the solution $z : [0, \infty) \to \mathbb{R}$ to*

$$z' - 2p(t)z(t) - 2q(t) = 0, t > 0 \text{ and } z(0) = a^{-2}$$

satisfies $z(t) > 0$ for all $t \geq 0$. Then $x : [0, \infty) \to \mathbb{R}$ given by $x(t) = \frac{1}{\sqrt{z(t)}}$ is the unique solution to

$$z' + p(t)x(t) + q(t)x^3(t) = 0, t > 0 \text{ and } x(0) = a$$

Theorem 0.4 *Let (α, β) be an open interval in $\mathbb{R}$. Let $f : (\alpha, \beta) \times \mathbb{R} \to \mathbb{R}$ be continuous and Lipschitz in x uniformly over compact subsets of (α, β). Let $t_0 \in (\alpha, \beta)$. Then the initial value problem*

$$x'(t) = f(t, x(t)),$$

for $t \in (\alpha, \beta)$ and $x(t_0) = x_0 \in \mathbb{R}$ has a unique solution that is continuously differentiable in (α, β).