Due date: October 31st, 2013

1. Let $u : \mathbb{R}^2 \to \mathbb{R}$. Suppose $u \in C^2(\mathbb{R})$. Consider the transformation $x = r\cos(\theta)$ and $y = r\sin(\theta)$ such that $r > 0$, $-\pi \leq \theta \leq \pi$. Let $v : [0,\infty) \times [-\pi,\pi] \to \mathbb{R}$ be given by

$$v(r,\theta) = u(x,y)$$

with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial^2 r} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$

2. Let $d \geq 2$, $u : \mathbb{R}^d \to \mathbb{R}$. Suppose $u \in C^2(\mathbb{R}^d)$. Let

$$u(x) = v(\mid x \mid)$$

where $v : [0,\infty) \to \mathbb{R}$ and $v \in C^([0,\infty))$. Then show that

$$\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 v}{\partial r^2} + \frac{d-1}{r}\frac{\partial v}{\partial r}.$$

3. Let $D = \{x \in \mathbb{R}^2 : 1 <\mid x \mid< 2\}$. Solve the following Dirichlet problem.

$$\begin{aligned}\Delta u &= 0, \text{ if } x \in D\\ u(x) &= 0, \text{ if } \mid x \mid= 1\\ u(x) &= 1, \text{ if } \mid x \mid= 2,\ x_2 > 0\\ u(x) &= -1, \text{ if } \mid x \mid= 2,\ x_2 < 0\end{aligned}$$

(you may assume that the above problem has a unique solution).

4. Find $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial^2 x},\ t > 0, x \in \mathbb{R}.$$

with initial value

$$u(0,x) = T1_{-1<x<0} + S1_{0<x<1} + \frac{S+T}{2}1_{x=0} \tag{1}$$

(you may assume that the above problem has a unique bounded solution).

5. Justify the following statement: Let $D = \{x \in \mathbb{R}^2 :\mid x \mid\leq 1\}$. Solving the Dirichlet problem:

$$\begin{aligned}\Delta u &= 0, \text{ if } x \in D\\ &= g(\theta), \text{ if } \mid x \mid= 1, x = (1,\theta), -\pi \leq \theta < \pi\end{aligned}$$

ie equivalent to obtaining the fourier series expansion for g.

Correction from Hw 7:

3. Find $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial^2 x},\ \ t > 0, x \in \mathbb{R}.$$

with initial value $u(0, x) = 0$ and $\frac{\partial u}{\partial t}(0, x) = 4\cos(5x)$.

4. Find $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 u}{\partial^2 x} + \beta(t)\alpha(x),\ \ t > 0, x \in \mathbb{R}.$$

with initial value $u(0, x) = f(x)$ and $\frac{\partial u}{\partial t}(0, x) = g(x)$. Assume that f, g, α are linear functions and β is continuous.