Problems due:None

Due date: August 29th, 2013

1. Let $r > 0, k > 0$. Let $x_1 : [0, \infty) \to \mathbb{R}$ and $x_2 : [0, \infty) \to \mathbb{R}$ be the two solutions to the IVP

$$\frac{dx}{dt}(t) = r(1 - \frac{x(t)}{k})x(t),\ t > 0, \text{ and}$$

$x_1(0) = a$, $x_2(0) = b$. If $a > b > 0$ then show that $x^1(t) > x^2(t)$ for all $t > 0$. In addtion show that $x(t) = 0$ for all $t \geq 0$ is a solution[1] of the above ordinary differential equation if $x(0) = 0$.

2. Let $X, f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ be continuous with $X(0) = a \in \mathbb{R}^2$. Show that

$$\frac{dX}{dt}(t) = f(t, X(t)),\ \forall t > 0$$

if and only if

$$X(t) = a + \int_0^t f(s, X(s))ds,\ \forall t > 0.$$

3. Let $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function such that for each $M > 0$ there exists a constant $K_M > 0$ such that

$$f(t, x) - f(t, y) \leq K_M \mid x - y \mid$$

for all $x, y \in \mathbb{R}^2$ and $t \in [0, M]$. Let $a \in \mathbb{R}^2$ then there is a unique solution to

$$\frac{dX}{dt}(t) = f(t, X(t)),\ \forall t > 0$$

and $X(0) = a$.
Hint: Imitate the proof of the one-dimensional case with suitable modifications.

4. Let $A_{2\times 2}$ be a 2×2-matrix of real numbers. Consider the linear system

$$\frac{dX}{dt}(t) = AX(t),\ \forall t > 0.$$

Show that e^{tA} is a fundamental matrix for the linear system.

5. Let $a > 0$. Consider the ordinary differential equation

$$\frac{dx}{dt}(t) = \frac{1}{4} - ax + x^2,\ t > 0.$$

What are the values of a for which x will stay finite for all $t > 0$? Do the solutions approach an equilibrium ?

[1] Extra Credit: Can you show it is the unique solution ?