1. Using the code of the `lupiv` function in the NMM toolbox (directory −`linalg`) solve for x when $Ax = b$ when

$$A = \begin{bmatrix} 2 & 4 & -2 & -2 \\ 1 & 2 & 4 & -3 \\ -3 & -3 & 8 & -2 \\ -1 & 1 & 6 & -3 \end{bmatrix} \quad b = \begin{bmatrix} -4 \\ 5 \\ 7 \\ 7 \end{bmatrix}$$

Provide the commands used by you in your answer.

2. Using code of the `cholesky` function in the NMM toolbox (directory −`linalg`) solve for x when $Ax = b$ when

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

Provide the commands used by you in your answer.

3. Use the function `linefit` in the NMM toolbox and fit a line to the following data:

x	1	2	4	5
y	1	2	2	3

Calculate the value of the R^2 statistic and decide if this is a good fit or not. Further plot data and the line obtained on one grid using the plot command.

4. The following three files, `cucon1.dat, cucon2.dat, cucon3.dat` have data with measured values of thermal conductivity k versus temperature T for three different samples of copper. The mathematical model for k vs T is widely believed to be

$$k(T) = \frac{1}{\frac{c_1}{T} + c_2 T^2}$$

Let $\gamma(T) = \frac{1}{k(T)}$. Use `fitnorm` and `cuconBasis1` to find the best possible c_1 and c_2 for γ versus T. Suppose we tweak the model to be

$$\gamma(T) = \frac{c_1}{T} + c_2 T^2 + c_3 T,$$

using fitnorm can you find the best possible fit. Compare the above two plots.

Due Date: March 25th, 2010
Problems to be turned in: 1,3,4,7.

1. Starting with the code `GEshow` in the NMM toolbox develop a `GErectangular` function that performs Gaussian elimination *only* for a $m \times n$ matrix. The function should return $\tilde{A}$, the triangularised augmented matrix.

   Bonus Question[1] : Write a general program in OCTAVE, `GEgeneral`, which solves a linear $m \times n$ system of equations.

2. Centrifugal pumps are common devices used to move liquid through piping systems. The key question there is to determine the pressure head h of the pump given q the flow rate. Start with the model specified in `pumpcurve` code in the NMM toolbox.

   (a) Consider q and h from the following table:

q(m^3/s)	0.0001	0.00025	0.0008	0.001	0.0014
h(m)	115	114.2	110	105.5	92.5

   Using the first three data points, write down the equation you get between h and q.

   (b) Modify the `pumpcurve` function to accept q and h vectors of arbitrary length as input. Using all data points (above) except the fourth, use your function to find the cubic polynomial interpolant. Also find the condition number of A. Let c be the coefficents of the polynomial.

   (c) Replace the second point to 114 from 114.2. Do as in previous part to get $\tilde{c}$.

   (d) Find the relative difference vector $d = \frac{\tilde{c}-c}{c}$ for all i.

   (e) Plot h vs q, and find the largest difference in the value of h from 100 points between $\min(q)$ and $\max q$ in both the cases.

   (f) Discuss the practical significance on the pertubation of h values on the coefficient c and the values of h obtained by the interpolation function.

3. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$.

   (a) Compute the condition number of A using the 1 norm or ∞ norm.

   (b) Deduce that the matrix is ill-conditioned for small ϵ.

   (c) Let $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\delta b = \begin{bmatrix} 0 \\ \epsilon \end{bmatrix}$ Show that

   $$\frac{\| \delta x \|}{\| x \|} = \kappa(A) \frac{\| \delta b \|}{\| b \|}$$

4. The function $y = \frac{x}{c_1 x + c_2}$ can be transformed into a linear relationship $z = c_1 w + c_2$ with the change of variable $z = \frac{1}{y}$, $w = \frac{1}{x}$. Write an `xlinxFit` function that calls `linefit` to fit data to $y = \frac{x}{c_1 + c_2 x}$. Test your function by fitting the following sets of data:

x	2.2500	2.5417	2.8333	3.1250	3.4167	3.7083	4.000
y	2.8648	1.936	1.0823	0.8842	0.7677	0.6910	0.6366

[1] Eventual Prize: Carrot Cake at end of semester

x	0.7000	1.0714	1.4429	1.8143	2.1857	2.5571	2.9286	3.3000
y	-0.1714	-0.3673	-0.8243	-3.1096	3.7463	1.4610	1.0039	0.8080

5. Consider the following table:

x	y
0.24	19.9
0.52	28.8
1.93	48.7
3.26	59.0
15.0	106.5

   (a) Construct a new table with $\log_{10} x$ and $\log_{10} y$.

   (b) Assume that data in your table is close to satistfying

$$\log_{10} y = \alpha \log_{10} x + \beta$$

   (c) Construct and solve the normal equations.

   (d) Plot the transformed data and the least-square line on the same axes.

   (e) Calculate the R^2 statistic. Plot the residual r as function of x.

6. Write a function to fit data to $y = c_1x^5 + c_2x^3 + c_3x + c_4$ without calling `polyfit, fitnorm` or `fitqr`. Your function should take two column vectors `x` and `y` as input and return the coefficient vector. Test your solution with the following data:

x	-3.0000	-1.8571	-0.7143	0.4286	1.5714	2.7143	3.8571	5.0000
y	-4.9262	-4.4720	-3.3136	-2.9396	-2.6697	-1.3906	-0.8669	-3.3823

7. **A QR Algorithm** Let c, s be two real numbers such that $c^2 + s^2 = 1$. For each $1 \leq i < j \leq n$, define the Q_{ij} as the matrix whose entries are given by:

$$q_{kl} = \begin{cases} 1 & \text{if } j \neq k = l \neq i \\ c & \text{if } k = l = j \text{ or } k = l = i \\ -s & \text{if } k = i, l = j \text{ or } k = j, l = i \\ 0 & \text{otherwise} \end{cases}$$

   (a) Verify that Q_{ij} is an orthogonal matrix.

   (b) Let $x_{n\times 1}$ be given. Find c, s such that $c^2 + s^2 = 1$ and j-th entry of $Q_{ij}x$ is 0 (*Observe that all other row entries except the i-th row remain unchanged*).

   (c) Let $n = 2$ and $A_{2\times 2}$ be given a matrix of rank 2. Can you now construct an orthogonal matrix $Q_{2\times 2}$ such that $A = QR$ where $R_{2\times 2}$ is upper triangular.

   (d) Let $n = 3$ and $A_{3\times 3}$ be given a matrix of rank 3. Can you now construct an orthogonal matrix $Q_{3\times 3}$ such that $A = QR$ where $R_{3\times 3}$ is upper triangular ?.

   (e) Now for any general $A_{m\times n}$ of full column rank, construct an orthogonal matrix $Q_{n\times n}$ such that $A = QR$ where $R_{m\times n}$ matrix whose first n rows form an upper-triangular matrix and the last $m - n$ rows are 0.

8. Consider the 2×2 non-linear system given by

$$\begin{aligned} x_1 - x_2 - 3 &= 0 \\ x_1^2 - 20x_1 - x_2 + 5 &= 0 \end{aligned}$$

(a) Identify $A(x), b(x)$ such that the above can be written as $A(x)x = b(x)$. Modify `demoSSub` to obtain a solution of the same. Use 10 iterations and find two initializing vectors that give you the two solutions.

(b) Identify f such that the above can be written as $f(x) = 0$. Modify `demoNewtonSys` to obtain a solution of the same. Use 10 iterations and find two initializing vectors that give you the two solutions.

9. Consider the matrices

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-6} \end{bmatrix}, A = \begin{bmatrix} 0.1 & 0.1 & 10^6 \\ 0.2 & -0.1 & 10^6 \\ 0.1 & 0.2 & 0 \end{bmatrix}, b = \begin{bmatrix} 0.2 + 10^6 \\ 0.1 + 10^6 \\ 0.3 \end{bmatrix}$$

(a) Using OCTAVE and the ∞ norm, compute (exactly) $\kappa(A)$. Decide whether the matrix A is ill-conditioned or not.

(b) Using the \ operator in OCTAVE, solve $Ax = b$.

(c) Perturb a_{13} to get another matrix $A + \delta A$ such that $\frac{\|\delta A\|}{\|A\|} \sim 10^{-6}$. Using the \ operator in OCTAVE, solve $(A + \delta A)\hat{x} = b$.

(d) Computer $\frac{\|x-\hat{x}\|}{\|x\|}$

(e) Let $\tilde{A} = AW$. Do parts (a) -(d) for $\tilde{A}$. Conclude that this matrix A was an "artificially" ill-conditioned matrix.

At the end of this chapter you should be able to

1. Describe the most efficient procedures for solving $Lx = b$ or $Ux = b$ when L is lower triangular and U is upper triangular.
2. Name the solution algorithm most commonly used for solving $Ax = b$.
3. Write the equation that defines the residual vector.
4. Describe the significance of $\kappa(A)$ on the reliability of the numerical solution to $Ax = b$.
5. Describe the reason for pivoting. Is pivoting a remedy for ill-conditioned systems?
6. Estimate the number of correct significant digits in the numerical solution to $Ax = b$ given values of ϵ_m and $\kappa(A)$.
7.
8. State conditions required for a successful LU factorization of A. Write (describe) a procedure for solving $Ax = b$ given an LU factorization of A.
9. State conditions required for a successful Cholesky factorization of A. Write (describe) a procedure for solving $Ax = b$ given a Cholesky factorization of A.
10. Use OCTAVE and the LU factorization of A to solve several systems of equations that have the same A and a sequence of different b.
11. Use OCTAVE and a Cholesky factorization of A to solve several systems of equations that have the same A and a sequence of different b.
12. List the order of magnitude work estimates for Gaussian elimination with back substitution, LU factorization, and Cholesky factorization.
13. Use the `linefit` function in the NMM Toolbox to obtain a least squares line fit to data and to linearized transformations of $y = c_1 e^{c_2 x}$ and $y = c_1 x^{c_2}$.
14. Use the built-in `polyfit` function to obtain a least squares fit to a polynomial in x. Use the built-in `polyval` function to evaluate the polynomial obtained from `polyfit` at any x.
15. Plot a comparison of a least squares fit and the data used to obtain the fit.
16. Use the `fitnorm` and `fitQR` functions from the NMM Toolbox to obtain least squares fits to linear combinations of arbitrary basis functions.
17. Assign the elements of the matrix `A` of the overdetermined system for a given choice of basis functions. Given this matrix and a vector of y data, obtain the coefficients of the fit by solving the normal equations.
18. Given the matrix of the overdetermined system for the fit, and a vector of y data values, use the \ operator to obtain the coefficients that minimizes $\| r \|_2$ for the overdetermined system.