Due: Wednesday, August 22, 2012
Problem to be turned in : 2, 5, 10, 12.

1. Let S be a subset of $\mathbb{R}$. Suppose $u \in S$ is an upper bound for S then find the $\sup(S)$.

2. Let $S \subset \mathbb{R}$, be bounded above. Show that $u = \sup(S)$ if and only if for every $\epsilon > 0$ there is a $s \equiv s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.

3. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let
$$A + B := \{a + b : a \in A, b \in B\}, \quad A \cdot B = \{a.b : a \in A, b \in B\}.$$
Prove that
   (a) $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$
   (b) $\inf(A + B) = \inf(A) + \inf(B)$.
   (c) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$. Let $E \subset \mathbb{R}$. If $f(x) \leq g(x)$ for all x in E, show that
$$\sup_{x \in E} f(x) \leq \sup_{x \in E} g(x).$$
How do things change if sup is replaced by inf?

5. Find the lower bounds, infimum, upper bounds and supremum (if any) for the following sets in the extended real number system $\mathbb{R}$:
$$A = (3, 4] \cup \{100\}, B = \{\frac{1}{2^n} + \frac{1}{3^n} : n \in \mathbb{N}\}$$
$$C = \{x \in \mathbb{R} : x = -9 + \frac{(-1)^n}{n}, n \geq 1\},$$
and
$$D = \{x \in \mathbb{R} : x = -15 + \frac{1}{n}, n \geq 1\} \cup \{x \in \mathbb{R} : x = 15 - \frac{1}{n}, n \geq 1\},$$

6. Let $f : \mathbb{R} \to \mathbb{R}$. be defined by $f(x) = x^2$. Then
   (a) Find $f(E)$, where $E = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$.
   (b) If $G := f(E)$, then find $f^{-1}(G)$ and $f(f^{-1}(G))$. (*Observe that* $f(f^{-1}(G)) = G$ *but* $f^{-1}(f(E)) \neq E$.)

7. Let $f : A \to B$. Let G be subset of B and $H \subset A$. Show that
$$f(f^{-1}(G)) \subset G, f^{-1}(f(H)) \supset H.$$

8. If $f : A \to B$ is injective and $g : B \to C$ is injective, then the composition $g \circ f : A \to C$ is injective.

9. If $a, b \in \mathbb{R}$ and $a < b$, then show that $a < \frac{a+b}{2} < b$.
10. If $a \in \mathbb{R}$ such that $0 \leq a < \epsilon$ for every $\epsilon > 0$, then show that $a = 0$.
11. Suppose $I_n = (10, 10 + \frac{1}{n})$ then show that $\cap_{n=1}^{\infty} I_n = \emptyset$.
12. Show that $\mathbb{R}$ is uncountable.
13. Let $A \subset B$. Let $f : A \to B$ be a bijection. Then conclude that either $\text{card}(A) = \text{card}(B)$ or A is infinite.