Due: Wednesday, November 14th, 2012
Problems to be turned in: 1,2

1. Let $-\infty \leq a < b \leq \infty$ and $f : (a, b) \to \mathbb{R}$ be a differentiable function.
   (a) If $c \in (a, b)$ is a local maximum of f (i.e. there is a $\delta > 0$ such that $f(x) \leq f(c)$ whenever $| x - c |< \delta$ and $x \in (a, b)$.) then $f'(c) = 0$.
   (b) f is increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.
   (c) Suppose $f'(c) = 0$ for all $c \in (a, b)$. From first principles, show that f is a constant function.
   (d) Can you construct a $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ but f is not monotonic in any neighbourhood of 0 ?
2. Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that for some $c \in (a, b)$, $f(c) = g(c)$ and $f'(c) < g'(c)$. Prove that there exists $\delta > 0$ such that $f(x) < g(x)$ for all $x \in (c, c + \delta)$, and $f(x) > g(x)$ for all $x \in (c - \delta, c)$.
3. Let $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $g(a) < g(b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$. Prove that g is strictly increasing on $[a, b]$.
4. Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable at some $c \in (a, b)$ with $f'(c) > 0$. Does this imply that f is strictly increasing, or increasing, on an interval $(c - \delta, c + \delta)$ for some $\delta > 0$? If true, then prove it, otherwise construct a counter-example.
5. Use the Mean Value Theorem to prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
6. If $h(x) = -1$ for $x < 0$ and $h(x) = 1$ for $x \geq 0$, prove that there exists no $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = h(x)$ for all $x \in \mathbb{R}$. (Hint: We have used the MVT to prove that the anti-derivative is unique up to a constant shift. Namely, if $g'(x) = f'(x)$ on an open interval (a, b), then $g(x) = f(x) + C$ for some constant C.)
7. Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$. If $f(0) = 1$ and $f(1) = f(2) = 0$, then show that
   (i) There exists $c_1 \in (0, 1)$ such that $f'(c_1) = -1$.
   (ii) There exists $c_2 \in (1, 2)$ such that $f'(c_2) = 0$.
   (iii) There exists $c_3 \in (0, 2)$ such that $f'(c_3) = -0.3$.
8. Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b). Show by an example that even if $\lim_{x\to a^+} f'(x)$ exists, f may not be differentiable at a. However, if f is furthermore assumed to be continuous at a, then $\lim_{x\to a} f'(x) = A$ implies $f'(a)$ exists and equals A.