Due: Wed november 7th, 2012
Problems to be turned in: 1,2

1. Let A be a countable subset of $\mathbb{R}$. Consider $p : A \to [0,1]$ such that $\sum_{n=1}^{\infty} p(x_n) = 1$ where $\{x_k : k \in \mathbb{N}\}$ is an enumeration of A. Define $F : \mathbb{R} \to [0,1]$ by $F(x) = \sum_{x_n \leq x} p(x_n) \equiv \sum_{n=1}^{\infty} g^x(x_n)p(x_n)$.
   (a) Show that F is monotonically increasing.
   (b) Identify the discontinuity points of F and show that $F(x+) = F(x)$ for all $x \in \mathbb{R}$ (where $F(x+)$ is the notation for RHL at x).
   (c) Show that $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.
   (d) By choosing a suitable A and p construct an example of a monotonically increasing function whose points of discontinuity are not isolated.

2. Find the continuity points of $f : \mathbb{R} \to \mathbb{R}$, when f is given by: (a) $f(x) = \lfloor x \rfloor$ (i.e. greatest integer less than or equal to x), (b) $f(x) = x\lfloor x \rfloor$, and (c) $f(x) = x - \lfloor x \rfloor$.

3. Show that $g(x) = \sqrt{x}$ is a uniformly continuous function on $[0,1]$ but is not a Lipschitz function.

4. Let $a, b \in \mathbb{R}$ and $I = (a,b)$. Let $f : I \to \mathbb{R}$ be a continuous monotonically (strictly) increasing function. Show that $f(I)$ is also an open interval. Is this interval always bounded ?

5. Rudin Exercise 6, chapter 1.

6. Let $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$
   Show that f is continuous at 0 and LHD (left hand derivative) and RHD(Right hand derivative) of f does not exist at 0.

7. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that
$$| f(x) - f(y) | \leq (x-y)^2$$
   Show that f is a constant function.

8. Let $f : \mathbb{R} \to \mathbb{R}$. Show that $\lim_{x\to 0} f(x)$ exists **if and only if** $\lim_{x\downarrow 0} f(x) := f(0+) = f(0-) := \lim_{x\uparrow 0} f(x)$.